The zeta function of $\mathfrak{tr}_7(\mathbb{Z})$ counting ideals

1 Introduction

$\mathfrak{tr}_7(\mathbb{Z})$ is the Lie ring of upper-triangular 7×7 matrices over $\mathbb{Z}$.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{tr}_7(\mathbb{Z}),p}(s) &= \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(s-5)\zeta_p(s-6)\\ &\times \zeta_p(2s)^3\zeta_p(4s)\zeta_p(5s)^2\zeta_p(9s)\zeta_p(11s)\zeta_p(12s)\zeta_p(13s)\zeta_p(14s)\\ &\times \zeta_p(15s)\zeta_p(16s)\zeta_p(17s)\zeta_p(18s)\zeta_p(19s)\zeta_p(20s)\zeta_p(21s)\zeta_p(22s)\\ &\times \zeta_p(23s)\zeta_p(24s)\zeta_p(25s)\zeta_p(26s)\zeta_p(27s)W(p,p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned}&1 + 3Y^{2} + 5Y^{4} + 3Y^{5} + 7Y^{6} + 9Y^{7} + 13Y^{8} + 18Y^{9} + 25Y^{10} + 32Y^{11} + 44Y^{12}\\
&+ 56Y^{13} + 75Y^{14} + 94Y^{15} + 125Y^{16} + 153Y^{17} + 199Y^{18} + 242Y^{19} + 305Y^{20}\\
&+ 367Y^{21} + 459Y^{22} + 545Y^{23} + 673Y^{24} + 793Y^{25} + 958Y^{26} + 1124Y^{27}\\
&+ 1337Y^{28} + 1553Y^{29} + 1834Y^{30} + 2106Y^{31} + 2458Y^{32} + 2806Y^{33} + 3228Y^{34}\\
&+ 3656Y^{35} + 4172Y^{36} + 4668Y^{37} + 5290Y^{38} + 5867Y^{39} + 6573Y^{40} + 7245Y^{41}\\
&+ 8028Y^{42} + 8767Y^{43} + 9642Y^{44} + 10421Y^{45} + 11360Y^{46} + 12183Y^{47}\\
&+ 13136Y^{48} + 13963Y^{49} + 14921Y^{50} + 15683Y^{51} + 16609Y^{52} + 17279Y^{53}\\
&+ 18089Y^{54} + 18627Y^{55} + 19271Y^{56} + 19582Y^{57} + 20023Y^{58} + 20038Y^{59}\\
&+ 20192Y^{60} + 19882Y^{61} + 19663Y^{62} + 18961Y^{63} + 18352Y^{64} + 17163Y^{65}\\
&+ 16125Y^{66} + 14444Y^{67} + 12905Y^{68} + 10732Y^{69} + 8700Y^{70} + 5995Y^{71}\\
&+ 3517Y^{72} + 305Y^{73} - 2612Y^{74} - 6241Y^{75} - 9546Y^{76} - 13535Y^{77}\\
&- 17095Y^{78} - 21361Y^{79} - 25071Y^{80} - 29441Y^{81} - 33196Y^{82} - 37522Y^{83}\\
&- 41121Y^{84} - 45290Y^{85} - 48557Y^{86} - 52361Y^{87} - 55180Y^{88} - 58427Y^{89}\\
&- 60607Y^{90} - 63191Y^{91} - 64544Y^{92} - 66322Y^{93} - 66778Y^{94} - 67583Y^{95}\\
&- 67068Y^{96} - 66871Y^{97} - 65267Y^{98} - 64071Y^{99} - 61396Y^{100} - 59142Y^{101}\\
&- 55484Y^{102} - 52239Y^{103} - 47622Y^{104} - 43560Y^{105} - 38095Y^{106}\end{aligned}$$

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\begin{aligned}
&- 33306Y^{107} - 27241Y^{108} - 21857Y^{109} - 15362Y^{110} - 9666Y^{111} - 2883Y^{112} \\
&+ 2883Y^{113} + 9666Y^{114} + 15362Y^{115} + 21857Y^{116} + 27241Y^{117} + 33306Y^{118} \\
&+ 38095Y^{119} + 43560Y^{120} + 47622Y^{121} + 52239Y^{122} + 55484Y^{123} \\
&+ 59142Y^{124} + 61396Y^{125} + 64071Y^{126} + 65267Y^{127} + 66871Y^{128} \\
&+ 67068Y^{129} + 67583Y^{130} + 66778Y^{131} + 66322Y^{132} + 64544Y^{133} \\
&+ 63191Y^{134} + 60607Y^{135} + 58427Y^{136} + 55180Y^{137} + 52361Y^{138} \\
&+ 48557Y^{139} + 45290Y^{140} + 41121Y^{141} + 37522Y^{142} + 33196Y^{143} \\
&+ 29441Y^{144} + 25071Y^{145} + 21361Y^{146} + 17095Y^{147} + 13535Y^{148} \\
&+ 9546Y^{149} + 6241Y^{150} + 2612Y^{151} - 305Y^{152} - 3517Y^{153} - 5995Y^{154} \\
&- 8700Y^{155} - 10732Y^{156} - 12905Y^{157} - 14444Y^{158} - 16125Y^{159} \\
&- 17163Y^{160} - 18352Y^{161} - 18961Y^{162} - 19663Y^{163} - 19882Y^{164} \\
&- 20192Y^{165} - 20038Y^{166} - 20023Y^{167} - 19582Y^{168} - 19271Y^{169} \\
&- 18627Y^{170} - 18089Y^{171} - 17279Y^{172} - 16609Y^{173} - 15683Y^{174} \\
&- 14921Y^{175} - 13963Y^{176} - 13136Y^{177} - 12183Y^{178} - 11360Y^{179} \\
&- 10421Y^{180} - 9642Y^{181} - 8767Y^{182} - 8028Y^{183} - 7245Y^{184} - 6573Y^{185} \\
&- 5867Y^{186} - 5290Y^{187} - 4668Y^{188} - 4172Y^{189} - 3656Y^{190} - 3228Y^{191} \\
&- 2806Y^{192} - 2458Y^{193} - 2106Y^{194} - 1834Y^{195} - 1553Y^{196} - 1337Y^{197} \\
&- 1124Y^{198} - 958Y^{199} - 793Y^{200} - 673Y^{201} - 545Y^{202} - 459Y^{203} \\
&- 367Y^{204} - 305Y^{205} - 242Y^{206} - 199Y^{207} - 153Y^{208} - 125Y^{209} - 94Y^{210} \\
&- 75Y^{211} - 56Y^{212} - 44Y^{213} - 32Y^{214} - 25Y^{215} - 18Y^{216} - 13Y^{217} - 9Y^{218} \\
&- 7Y^{219} - 3Y^{220} - 5Y^{221} - 3Y^{223} - Y^{225}.
\end{aligned}
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$\zeta^{\triangleleft}_{\mathfrak{tr}_7(\mathbb{Z})}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{tr}_7(\mathbb{Z}),p}(s)\Big|_{p\to p^{-1}} = p^{21-134s}\zeta^{\triangleleft}_{\mathfrak{tr}_7(\mathbb{Z}),p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{tr}_7(\mathbb{Z})}(s)$ is 7, with a simple pole at $s = 7$.

5 Ghost zeta function

This zeta function is its own ghost.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{tr}_7(\mathbb{Z})}(s)$ has natural boundary at $\Re(s) = 0$.