The zeta function of $\mathfrak{tr}_5(\mathbb{Z})$ counting ideals

1 Introduction

$\mathfrak{tr}_5(\mathbb{Z})$ is the Lie ring of upper-triangular 5×5 matrices over $\mathbb{Z}$.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{tr}_5(\mathbb{Z}),p}(s) &= \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(2s)^3\zeta_p(5s)^2\zeta_p(8s) \\ &\times \zeta_p(9s)\zeta_p(11s)\zeta_p(12s)\zeta_p(13s)\zeta_p(14s)W(p,p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned}&1 + Y^2 + Y^4 + Y^5 + Y^6 + Y^7 + 2Y^8 + 2Y^9 + Y^{10} + 2Y^{11} + 2Y^{12} + Y^{13} + Y^{14} \\ &+ Y^{15} + Y^{16} + Y^{17} - Y^{20} - Y^{21} - Y^{22} - Y^{23} - Y^{24} - 2Y^{25} - 2Y^{26} - Y^{27} \\ &- 2Y^{28} - 2Y^{29} - Y^{30} - Y^{31} - Y^{32} - Y^{33} - Y^{35} - Y^{37}.\end{aligned}$$

$\zeta^{\triangleleft}_{\mathfrak{tr}_5(\mathbb{Z})}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{tr}_5(\mathbb{Z}),p}(s)\Big|_{p\to p^{-1}} = -p^{10-51s}\zeta^{\triangleleft}_{\mathfrak{tr}_5(\mathbb{Z}),p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{tr}_5(\mathbb{Z})}(s)$ is 5, with a simple pole at $s = 5$.

5 Ghost zeta function

This zeta function is its own ghost.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{tr}_5(\mathbb{Z})}(s)$ has natural boundary at $\Re(s) = 0$.