The zeta function of $\mathfrak{tr}_4(\mathbb{Z})$ counting ideals

1 Introduction

$\mathfrak{tr}_4(\mathbb{Z})$ is the Lie ring of upper-triangular 4×4 matrices over $\mathbb{Z}$.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta^{\triangleleft}_{\mathfrak{tr}_4(\mathbb{Z}),p}(s) = \zeta_p(s)^2\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s)^2\zeta_p(5s)\zeta_p(8s)\zeta_p(9s) \times W(p, p^{-s})$$

where $W(X, Y)$ is

$$1 - Y + Y^2 - Y^3 + Y^4.$$

$\zeta^{\triangleleft}_{\mathfrak{tr}_4(\mathbb{Z})}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{tr}_4(\mathbb{Z}),p}(s)\Big|_{p\to p^{-1}} = p^{6-27s}\zeta^{\triangleleft}_{\mathfrak{tr}_4(\mathbb{Z}),p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{tr}_4(\mathbb{Z})}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

This zeta function is its own ghost.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{tr}_4(\mathbb{Z})}(s)$ has meromorphic continuation to $\mathbb{C}$.