

The zeta function of $\mathrm{tr}_2(\mathbb{Z})$ counting all subrings

1 Introduction

$\mathrm{tr}_2(\mathbb{Z})$ is the Lie ring of upper-triangular 2×2 matrices over \mathbb{Z} .

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta_{\mathrm{tr}_2(\mathbb{Z}),p}(s) = \zeta_p(s)\zeta_p(s-1)^2\zeta_p(2s-2)\zeta_p(2s-1)^{-1}.$$

$\zeta_{\mathrm{tr}_2(\mathbb{Z})}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{\mathrm{tr}_2(\mathbb{Z}),p}(s)\Big|_{p \rightarrow p^{-1}} = -p^{3-3s}\zeta_{\mathrm{tr}_2(\mathbb{Z}),p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{\mathrm{tr}_2(\mathbb{Z})}(s)$ is 2, with a double pole at $s = 2$.

5 Ghost zeta function

This zeta function is its own ghost.

6 Natural boundary

$\zeta_{\mathrm{tr}_2(\mathbb{Z})}(s)$ has meromorphic continuation to \mathbb{C} .