

# The zeta function of $\mathrm{tr}_2(\mathbb{Z})$ counting ideals

## 1 Introduction

$\mathrm{tr}_2(\mathbb{Z})$  is the Lie ring of upper-triangular  $2 \times 2$  matrices over  $\mathbb{Z}$ .

## 2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta_{\mathrm{tr}_2(\mathbb{Z}),p}^{\triangleleft}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(2s).$$

$\zeta_{\mathrm{tr}_2(\mathbb{Z})}^{\triangleleft}(s)$  is uniform.

## 3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{\mathrm{tr}_2(\mathbb{Z}),p}^{\triangleleft}(s) \Big|_{p \rightarrow p^{-1}} = -p^{1-4s} \zeta_{\mathrm{tr}_2(\mathbb{Z}),p}^{\triangleleft}(s).$$

## 4 Abscissa of convergence and order of pole

The abscissa of convergence of  $\zeta_{\mathrm{tr}_2(\mathbb{Z})}^{\triangleleft}(s)$  is 2, with a simple pole at  $s = 2$ .

## 5 Ghost zeta function

This zeta function is its own ghost.

## 6 Natural boundary

$\zeta_{\mathrm{tr}_2(\mathbb{Z})}^{\triangleleft}(s)$  has meromorphic continuation to  $\mathbb{C}$ .