The zeta function of $\mathfrak{sl}_2(\mathbb{Z})$ counting ideals

1 Presentation

$\mathfrak{sl}_2(\mathbb{Z})$ has presentation

$$\langle f, e, h \mid [h, e] = 2e, [h, f] = -2f, [e, f] = h \rangle .$$

$\mathfrak{sl}_2(\mathbb{Z})$ is insoluble.

2 The local zeta function

The local zeta functions for Lie rings L additively isomorphic to $\mathbb{Z}^d$ for some d and with L/pL simple were calculated by Marcus du Sautoy. Indeed they are

$$\zeta^{\triangleleft}_{L,p}(s) = \zeta_p(ds).$$

This applies to $\mathfrak{sl}_2(\mathbb{Z})$ for $p > 2$, and so we have

$$\zeta^{\triangleleft}_{\mathfrak{sl}_2(\mathbb{Z}),p}(s) = \zeta_p(3s).$$

The case $p = 2$ requires separate attention. Luke Woodward has calculated that

$$\zeta^{\triangleleft}_{\mathfrak{sl}_2(\mathbb{Z}),2}(s) = \zeta_2(3s)(1 + 3.2^{-s} + 2^{-2s}).$$

3 Functional equation

For $p \neq 2$, the local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{sl}_2(\mathbb{Z}),p}(s)|_{p \to p^{-1}} = -p^{-3s}\zeta^{\triangleleft}_{\mathfrak{sl}_2(\mathbb{Z}),p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{sl}_2(\mathbb{Z})}(s)$ is $1/3$, with a simple pole at $s = 1/3$.

5 Ghost zeta function

The ghost zeta function is unknown since the zeta functions vary with the prime.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{sl}_2(\mathbb{Z})}(s)$ has meromorphic continuation to the whole of $\mathbb{C}$.