

# The zeta function of $\mathfrak{p}_4$ counting all subgroups

## 1 Presentation

$\mathfrak{p}_4$  has presentation

$$\langle x, y, r \mid [x, y], r^4, y^r = x^{-1}, x^r = y \rangle.$$

## 2 The zeta function itself

The zeta function was calculated by du Sautoy, McDermott and Smith. It is

$$\zeta_{\mathfrak{p}_4}(s) = \zeta(s-1)L(s-1, \chi_4) + 4^{-s}\zeta(s)\zeta(s-1) + 2^{-s}\zeta(s-1)\zeta(s-2),$$

where  $\chi_4$  is defined by

$$\chi_4(a) = \begin{cases} 1 & \text{if } a \equiv 1 \pmod{4} \\ -1 & \text{if } a \equiv -1 \pmod{4} \\ 0 & \text{otherwise} \end{cases},$$

and  $L(s, \chi_4)$  is the Dirichlet  $L$ -function of  $\chi_4$ ,

$$L(s, \chi_4) = \sum_{n=1}^{\infty} \chi_4(n)n^{-s}.$$

## 3 Abscissa of convergence and order of pole

The abscissa of convergence of  $\zeta_{\mathfrak{p}_4}(s)$  is 3, with a simple pole at  $s = 3$ . Since this group is a finite extension of a free abelian group, its zeta function has meromorphic continuation to  $\mathbb{C}$ .