The zeta function of p4 counting all subgroups

1 Presentation

p4 has presentation

$$\left\langle x, y, r \mid [x, y], r^4, y^r = x^{-1}, x^r = y \right\rangle .$$

2 The zeta function itself

The zeta function was calculated by du Sautoy, McDermott and Smith. It is

$$\zeta_{\mathbf{p4}}(s) = \zeta(s-1)L(s-1, \chi_4) + 4^{-s}\zeta(s)\zeta(s-1) + 2^{-s}\zeta(s-1)\zeta(s-2),$$

where χ_4 is defined by

$$\chi_4(a) = \begin{cases} 1 & \text{if } a \equiv 1(\pmod 4) \\ -1 & \text{if } a \equiv -1(\pmod 4) \\ 0 & \text{otherwise} \end{cases},$$

and $L(s, \chi_4)$ is the Dirichlet L-function of χ_4,

$$L(s, \chi_4) = \sum_{n=1}^{\infty} \chi_4(n) n^{-s}.$$

3 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{\mathbf{p4}}(s)$ is 3, with a simple pole at $s = 3$. Since this group is a finite extension of a free abelian group, its zeta function has meromorphic continuation to $\mathbb{C}$.