The zeta function of $\mathfrak{g}_{6,7}$ counting ideals

1 Presentation

$\mathfrak{g}_{6,7}$ has presentation

$$\langle x_1, x_2, x_3, x_4, x_5, x_6 \mid [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_6 \rangle .$$

$\mathfrak{g}_{6,7}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{6,7},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-4)\zeta_p(4s-3)\zeta_p(5s-5)\zeta_p(5s-6) \\ & \times \zeta_p(6s-6)\zeta_p(7s-7)W(p,p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned}&1 + X^3Y^3 - X^3Y^5 - 2X^6Y^7 - X^7Y^8 - X^9Y^9 - X^{10}Y^{10} + X^9Y^{11} - X^{10}Y^{11} \\ &+ 2X^{10}Y^{12} + X^{12}Y^{14} + X^{13}Y^{14} + X^{13}Y^{15} + X^{16}Y^{16} - X^{16}Y^{19} - X^{19}Y^{21}.\end{aligned}$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{6,7}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies no functional equation.

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{g}_{6,7}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-4)\zeta_p(4s-3)\zeta_p(5s-5)\zeta_p(5s-6)\zeta_p(6s-6) \\ &\times \zeta_p(7s-7)W_1(p,p^{-s})W_2(p,p^{-s})\end{aligned}$$

where

$$W_1(X,Y) = 1 + X^3Y^3 - X^9Y^9 - X^{10}Y^{10} + X^{16}Y^{16},$$
$$W_2(X,Y) = 1 - X^3Y^5.$$

The ghost is unfriendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{g}_{6,7}}(s)$ has a natural boundary at $\Re(s) = 1$, and is of type I.