The zeta function of $\mathfrak{g}_{6,6}$ counting ideals

1 Presentation

$\mathfrak{g}_{6,6}$ has presentation

$$\langle x_1, x_2, x_3, x_4, x_5, x_6 \mid [x_1, x_2] = x_6, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5 \rangle .$$

$\mathfrak{g}_{6,6}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{6,6},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-4)\zeta_p(5s-5)\zeta_p(5s-6)\zeta_p(6s-6) \\ & \times \zeta_p(7s-8)\zeta_p(9s-11)W(p, p^{-s})\end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned}& 1 + X^3Y^3 - X^6Y^7 - X^8Y^8 - X^9Y^9 - 2X^{11}Y^{10} - X^{14}Y^{12} + X^{14}Y^{14} \\ & - X^{15}Y^{14} + X^{15}Y^{15} + X^{17}Y^{16} + X^{17}Y^{17} + X^{19}Y^{17} + X^{20}Y^{19} + X^{21}Y^{19} \\ & - X^{21}Y^{20} + X^{22}Y^{20} - X^{25}Y^{24} - X^{28}Y^{26}.\end{aligned}$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{6,6}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies no functional equation.

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{g}_{6,6}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}& \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-4)\zeta_p(5s-5)\zeta_p(5s-6)\zeta_p(6s-6)\zeta_p(7s-8) \\ & \times \zeta_p(9s-11)W_1(p, p^{-s})W_2(p, p^{-s})\end{aligned}$$

where

$$W_1(X,Y) = 1 - X^{14}Y^{12},$$
$$W_2(X,Y) = -1 + X^5Y^5 + X^7Y^7 + X^8Y^8 - X^{14}Y^{14}.$$

The ghost is unfriendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{g}_{6,6}}(s)$ has a natural boundary at $\Re(s) = 7/6$, and is of type III.