The zeta function of $\mathfrak{g}_{6,17}$ counting ideals

1 Presentation

$\mathfrak{g}_{6,17}$ has presentation

$$\langle x_1, x_2, x_3, x_4, x_5, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_5] = x_6 \rangle .$$

$\mathfrak{g}_{6,17}$ has nilpotency class 4.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{6,17},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(4s-3)\zeta_p(6s-4)\zeta_p(7s-5) \\ & \times \zeta_p(9s-8)W(p, p^{-s})\end{aligned}$$

where $W(X, Y)$ is

$$1 - X^3Y^5 + X^4Y^5 - X^4Y^7 - X^7Y^9 + X^7Y^{11} - X^8Y^{11} + X^{11}Y^{16}.$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{6,17}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\left.\zeta^{\triangleleft}_{\mathfrak{g}_{6,17},p}(s)\right|_{p \to p^{-1}} = p^{15-16s}\zeta^{\triangleleft}_{\mathfrak{g}_{6,17},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{g}_{6,17}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(4s-3)\zeta_p(6s-4)\zeta_p(7s-5)\zeta_p(9s-8) \\ &\times W_1(p, p^{-s})W_2(p, p^{-s})W_3(p, p^{-s})\end{aligned}$$

where

$$W_1(X,Y) = 1 + X^4Y^5,$$
$$W_2(X,Y) = 1 - X^3Y^4,$$
$$W_3(X,Y) = -1 + X^4Y^7.$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{g}_{6,17}}(s)$ has a natural boundary at $\Re(s) = 4/5$, and is of type III.

7 Notes

This ideal zeta function is identical to that of $\mathfrak{g}_{6,15}$, though the Lie rings themselves are non-isomorphic.