

The zeta function of $\mathfrak{g}_{6,15}$ counting ideals

1 Presentation

$\mathfrak{g}_{6,15}$ has presentation

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6 \mid \begin{array}{l} [x_1, x_2] = x_3 + x_5, [x_1, x_3] = x_4, \\ [x_1, x_4] = x_6, [x_2, x_5] = x_6 \end{array} \right\rangle.$$

$\mathfrak{g}_{6,15}$ has nilpotency class 4.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta_{\mathfrak{g}_{6,15},p}^{\triangleleft}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(4s-3)\zeta_p(6s-4)\zeta_p(7s-5) \\ \times \zeta_p(9s-8)W(p, p^{-s})$$

where $W(X, Y)$ is

$$1 - X^3Y^5 + X^4Y^5 - X^4Y^7 - X^7Y^9 + X^7Y^{11} - X^8Y^{11} + X^{11}Y^{16}.$$

$\zeta_{\mathfrak{g}_{6,15}}^{\triangleleft}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{\mathfrak{g}_{6,15},p}^{\triangleleft}(s) \Big|_{p \rightarrow p^{-1}} = p^{15-16s} \zeta_{\mathfrak{g}_{6,15},p}^{\triangleleft}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{\mathfrak{g}_{6,15}}^{\triangleleft}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(4s-3)\zeta_p(6s-4)\zeta_p(7s-5)\zeta_p(9s-8) \\ \times W_1(p, p^{-s})W_2(p, p^{-s})W_3(p, p^{-s})$$

where

$$\begin{aligned}W_1(X, Y) &= 1 + X^4Y^5, \\W_2(X, Y) &= 1 - X^3Y^4, \\W_3(X, Y) &= -1 + X^4Y^7.\end{aligned}$$

The ghost is friendly.

6 Natural boundary

$\zeta_{\mathfrak{g}_{6,15}}^{\triangleleft}(s)$ has a natural boundary at $\Re(s) = 4/5$, and is of type III.

7 Notes

This ideal zeta function is identical to that of $\mathfrak{g}_{6,17}$, though the Lie rings themselves are non-isomorphic.