The zeta function of $\mathfrak{g}_{6,14}(\gamma)$ counting ideals

1 Presentation

For $\gamma \in \mathbb{Z}$, $\gamma \neq 0$, $\mathfrak{g}_{6,14}(\gamma)$ has presentation

$$\langle x_1, x_2, x_3, x_4, x_5, x_6 \mid [x_1, x_3] = x_4, [x_1, x_4] = x_6, [x_2, x_3] = x_5, [x_2, x_5] = \gamma x_6 \rangle .$$

$\mathfrak{g}_{6,14}$ has nilpotency class 3.

2 The local zeta function

For all primes p not dividing γ, the local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{6,14(\gamma)},p}(s) &= \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(3s-4)\zeta_p(5s-6)\zeta_p(6s-3) \\ &\quad \times \zeta_p(7s-5)\zeta_p(6s-6)^{-1}\zeta_p(7s-3)^{-1}.\end{aligned}$$

The case where $p \mid \gamma$ has not been calculated.

$\zeta^{\triangleleft}_{\mathfrak{g}_{6,14}}(s)$ is uniform.

3 Functional equation

If $p \nmid \gamma$, the local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,14}(\gamma),p}(s)|_{p \to p^{-1}} = p^{15-14s}\zeta^{\triangleleft}_{\mathfrak{g}_{6,14}(\gamma),p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{g}_{6,14}(\pm 1)}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

For $\gamma = \pm 1$, this zeta function is its own ghost.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{g}_{6,14}(\gamma)}(s)$ has meromorphic continuation to the whole of $\mathbb{C}$ for all γ. For $|\gamma| > 1$, this follows since all but finitely many factors are built up from local Riemann zeta functions.

7 Notes

Some of the analytic properties of $\zeta^{\triangleleft}_{\mathfrak{g}_{6,14}(\gamma)}(s)$ for $|\gamma| > 1$ are not known, due to the fact that the local factors where $p \mid \gamma$ have not been calculated. However, it can be shown that if $p \mid \gamma$, $\zeta^{\triangleleft}_{\mathfrak{g}_{6,14}(\gamma),p}(s)$ depends only on the power of p dividing γ.

This family of Lie rings are all non-isomorphic over $\mathbb{Z}$, but $\zeta^{\triangleleft}_{\mathfrak{g}_{6,14}(\gamma),p}(s) = \zeta^{\triangleleft}_{\mathfrak{g}_{6,14}(-\gamma),p}(s)$, thus providing an infinite family of pairs of isospectral Lie rings.