The zeta function of $\mathfrak{g}_{6,13}$ counting ideals

1 Presentation

$\mathfrak{g}_{6,13}$ has presentation

$$\langle x_1, x_2, x_3, x_4, x_5, x_6 \mid [x_1, x_2] = x_5, [x_1, x_3] = x_4, [x_1, x_4] = x_6, [x_2, x_5] = x_6 \rangle .$$

$\mathfrak{g}_{6,13}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,13},p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-4)\zeta_p(5s-6)\zeta_p(6s-4)\zeta_p(7s-5) \times \zeta_p(9s-8)W(p,p^{-s})$$

where $W(X,Y)$ is

$$1 + X^3Y^3 - X^4Y^7 - X^7Y^9 - X^8Y^{10} - X^{11}Y^{12} + X^{12}Y^{16} + X^{15}Y^{19}.$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{6,13}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,13},p}(s)\Big|_{p\to p^{-1}} = p^{15-14s}\zeta^{\triangleleft}_{\mathfrak{g}_{6,13},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{g}_{6,13}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-4)\zeta_p(5s-6)\zeta_p(6s-4)\zeta_p(7s-5)\zeta_p(9s-8) \times W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})$$

where

$$
\begin{aligned}
W_1(X,Y) &= 1 + X^3Y^3, \\
W_2(X,Y) &= 1 - X^8Y^9, \\
W_3(X,Y) &= -1 + X^4Y^7.
\end{aligned}
$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{g}_{6,13}}(s)$ has a natural boundary at $\Re(s) = 1$, and is of type II.