The zeta function of $\mathfrak{g}_{6,12}$ counting all subrings

1 Presentation

$\mathfrak{g}_{6,12}$ has presentation

$$\langle x_1, x_2, x_3, x_4, x_5, x_6 \mid [x_1, x_3] = x_5, [x_1, x_5] = x_6, [x_2, x_4] = x_6 \rangle .$$

$\mathfrak{g}_{6,12}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta_{\mathfrak{g}_{6,12},p}(s) = {}& \zeta_p(s)\zeta_p(s-1)\zeta_p(s-3)\zeta_p(2s-5)\zeta_p(3s-5)\zeta_p(3s-6)\zeta_p(4s-8)\\ &\times \zeta_p(4s-9)\zeta_p(5s-12)\zeta_p(6s-12)\zeta_p(6s-13)\zeta_p(7s-16)\\ &\times W(p, p^{-s})\end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned}
&1 + X^2Y + 2X^4Y^2 - X^5Y^3 + 2X^6Y^3 - X^6Y^4 - X^7Y^4 + 4X^8Y^4 + X^9Y^4\\
&- 3X^8Y^5 - 5X^9Y^5 + X^{10}Y^5 + X^9Y^6 - X^{10}Y^6 - 3X^{11}Y^6 + 3X^{12}Y^6\\
&- X^{13}Y^6 + X^{11}Y^7 - 5X^{12}Y^7 - 8X^{13}Y^7 - 2X^{14}Y^7 - 3X^{15}Y^7 + 5X^{13}Y^8\\
&+ X^{14}Y^8 - X^{15}Y^8 - 2X^{16}Y^8 - 6X^{17}Y^8 + 2X^{15}Y^9 - 2X^{16}Y^9 + X^{17}Y^9\\
&- 2X^{18}Y^9 - 7X^{19}Y^9 - 2X^{20}Y^9 - X^{21}Y^9 + 4X^{17}Y^{10} + 3X^{18}Y^{10} + 8X^{19}Y^{10}\\
&- X^{20}Y^{10} - 4X^{21}Y^{10} - X^{23}Y^{10} - X^{18}Y^{11} + X^{20}Y^{11} + 11X^{21}Y^{11}\\
&+ X^{22}Y^{11} - 4X^{23}Y^{11} - 4X^{24}Y^{11} - 2X^{25}Y^{11} + 2X^{22}Y^{12} + 13X^{23}Y^{12}\\
&+ 8X^{24}Y^{12} + 8X^{25}Y^{12} + X^{26}Y^{12} - X^{27}Y^{12} - 3X^{23}Y^{13} - 2X^{24}Y^{13}\\
&+ 8X^{25}Y^{13} + 3X^{26}Y^{13} + 5X^{27}Y^{13} + X^{29}Y^{13} - 3X^{25}Y^{14} - 2X^{26}Y^{14}\\
&+ 6X^{27}Y^{14} + 8X^{28}Y^{14} + 13X^{29}Y^{14} + 3X^{30}Y^{14} + 2X^{31}Y^{14} - 5X^{27}Y^{15}\\
&- 5X^{28}Y^{15} - 4X^{29}Y^{15} - 3X^{30}Y^{15} + 9X^{31}Y^{15} + 6X^{32}Y^{15} + 4X^{33}Y^{15}\\
&- 2X^{29}Y^{16} - 3X^{30}Y^{16} - 8X^{31}Y^{16} - 5X^{32}Y^{16} + 6X^{33}Y^{16} + 2X^{34}Y^{16}\\
&+ 6X^{35}Y^{16} + 2X^{36}Y^{16} - 2X^{31}Y^{17} - X^{32}Y^{17} - 11X^{33}Y^{17} - 11X^{34}Y^{17}\\
&- X^{36}Y^{17} + 4X^{37}Y^{17} + X^{38}Y^{17} - 12X^{35}Y^{18} - 11X^{36}Y^{18} - 8X^{37}Y^{18}\\
&- 6X^{38}Y^{18} + 6X^{39}Y^{18} + 2X^{40}Y^{18} + 2X^{35}Y^{19} + 6X^{36}Y^{19} - 6X^{37}Y^{19}\\
&- 8X^{38}Y^{19} - 11X^{39}Y^{19} - 12X^{40}Y^{19} + X^{37}Y^{20} + 4X^{38}Y^{20} - X^{39}Y^{20}\\
&- 11X^{41}Y^{20} - 11X^{42}Y^{20} - X^{43}Y^{20} - 2X^{44}Y^{20} + 2X^{39}Y^{21} + 6X^{40}Y^{21}
\end{aligned}$$

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\begin{aligned}
&+2X^{41}Y^{21}+6X^{42}Y^{21}-5X^{43}Y^{21}-8X^{44}Y^{21}-3X^{45}Y^{21}-2X^{46}Y^{21}\\
&+4X^{42}Y^{22}+6X^{43}Y^{22}+9X^{44}Y^{22}-3X^{45}Y^{22}-4X^{46}Y^{22}-5X^{47}Y^{22}\\
&-5X^{48}Y^{22}+2X^{44}Y^{23}+3X^{45}Y^{23}+13X^{46}Y^{23}+8X^{47}Y^{23}+6X^{48}Y^{23}\\
&-2X^{49}Y^{23}-3X^{50}Y^{23}+X^{46}Y^{24}+5X^{48}Y^{24}+3X^{49}Y^{24}+8X^{50}Y^{24}\\
&-2X^{51}Y^{24}-3X^{52}Y^{24}-X^{48}Y^{25}+X^{49}Y^{25}+8X^{50}Y^{25}+8X^{51}Y^{25}\\
&+13X^{52}Y^{25}+2X^{53}Y^{25}-2X^{50}Y^{26}-4X^{51}Y^{26}-4X^{52}Y^{26}+X^{53}Y^{26}\\
&+11X^{54}Y^{26}+X^{55}Y^{26}-X^{57}Y^{26}-X^{52}Y^{27}-4X^{54}Y^{27}-X^{55}Y^{27}\\
&+8X^{56}Y^{27}+3X^{57}Y^{27}+4X^{58}Y^{27}-X^{54}Y^{28}-2X^{55}Y^{28}-7X^{56}Y^{28}\\
&-2X^{57}Y^{28}+X^{58}Y^{28}-2X^{59}Y^{28}+2X^{60}Y^{28}-6X^{58}Y^{29}-2X^{59}Y^{29}\\
&-X^{60}Y^{29}+X^{61}Y^{29}+5X^{62}Y^{29}-3X^{60}Y^{30}-2X^{61}Y^{30}-8X^{62}Y^{30}\\
&-5X^{63}Y^{30}+X^{64}Y^{30}-X^{62}Y^{31}+3X^{63}Y^{31}-3X^{64}Y^{31}-X^{65}Y^{31}\\
&+X^{66}Y^{31}+X^{65}Y^{32}-5X^{66}Y^{32}-3X^{67}Y^{32}+X^{66}Y^{33}+4X^{67}Y^{33}\\
&-X^{68}Y^{33}-X^{69}Y^{33}+2X^{69}Y^{34}-X^{70}Y^{34}+2X^{71}Y^{35}+X^{73}Y^{36}+X^{75}Y^{37}.
\end{aligned}
$$

$\zeta_{\mathfrak{g}_{6,12}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{\mathfrak{g}_{6,12},p}(s)\big|_{p\to p^{-1}} = p^{15-6s}\zeta_{\mathfrak{g}_{6,12},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{\mathfrak{g}_{6,12}}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$
\begin{aligned}
&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-3)\zeta_p(2s-5)\zeta_p(3s-5)\zeta_p(3s-6)\zeta_p(4s-8)\zeta_p(4s-9)\\
&\times\zeta_p(5s-12)\zeta_p(6s-12)\zeta_p(6s-13)\zeta_p(7s-16)W_1(p,p^{-s})W_2(p,p^{-s})\\
&\times W_3(p,p^{-s})W_4(p,p^{-s})W_5(p,p^{-s})
\end{aligned}
$$

where

$$W_1(X,Y) = 1 - X^{21}Y^9,$$
$$W_2(X,Y) = -1 + 2X^{15}Y^7,$$
$$W_3(X,Y) = 2 - X^{21}Y^{10},$$
$$W_4(X,Y) = -1 + X^9Y^5,$$
$$W_5(X,Y) = 1 - X^3Y^2 + X^9Y^6.$$

The ghost is unfriendly.

6 Natural boundary

$\zeta_{\mathfrak{g}_{6,12}}(s)$ has a natural boundary at $\Re(s) = 7/3$, and is of type III.