

The zeta function of $\mathfrak{g}_{6,12}$ counting all subrings

1 Presentation

$\mathfrak{g}_{6,12}$ has presentation

$$\langle x_1, x_2, x_3, x_4, x_5, x_6 \mid [x_1, x_3] = x_5, [x_1, x_5] = x_6, [x_2, x_4] = x_6 \rangle.$$

$\mathfrak{g}_{6,12}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta_{\mathfrak{g}_{6,12,p}}(s) &= \zeta_p(s)\zeta_p(s-1)\zeta_p(s-3)\zeta_p(2s-5)\zeta_p(3s-5)\zeta_p(3s-6)\zeta_p(4s-8) \\ &\quad \times \zeta_p(4s-9)\zeta_p(5s-12)\zeta_p(6s-12)\zeta_p(6s-13)\zeta_p(7s-16) \\ &\quad \times W(p, p^{-s}) \end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned} &1 + X^2Y + 2X^4Y^2 - X^5Y^3 + 2X^6Y^3 - X^6Y^4 - X^7Y^4 + 4X^8Y^4 + X^9Y^4 \\ &- 3X^8Y^5 - 5X^9Y^5 + X^{10}Y^5 + X^9Y^6 - X^{10}Y^6 - 3X^{11}Y^6 + 3X^{12}Y^6 \\ &- X^{13}Y^6 + X^{11}Y^7 - 5X^{12}Y^7 - 8X^{13}Y^7 - 2X^{14}Y^7 - 3X^{15}Y^7 + 5X^{13}Y^8 \\ &+ X^{14}Y^8 - X^{15}Y^8 - 2X^{16}Y^8 - 6X^{17}Y^8 + 2X^{15}Y^9 - 2X^{16}Y^9 + X^{17}Y^9 \\ &- 2X^{18}Y^9 - 7X^{19}Y^9 - 2X^{20}Y^9 - X^{21}Y^9 + 4X^{17}Y^{10} + 3X^{18}Y^{10} + 8X^{19}Y^{10} \\ &- X^{20}Y^{10} - 4X^{21}Y^{10} - X^{23}Y^{10} - X^{18}Y^{11} + X^{20}Y^{11} + 11X^{21}Y^{11} \\ &+ X^{22}Y^{11} - 4X^{23}Y^{11} - 4X^{24}Y^{11} - 2X^{25}Y^{11} + 2X^{22}Y^{12} + 13X^{23}Y^{12} \\ &+ 8X^{24}Y^{12} + 8X^{25}Y^{12} + X^{26}Y^{12} - X^{27}Y^{12} - 3X^{23}Y^{13} - 2X^{24}Y^{13} \\ &+ 8X^{25}Y^{13} + 3X^{26}Y^{13} + 5X^{27}Y^{13} + X^{29}Y^{13} - 3X^{25}Y^{14} - 2X^{26}Y^{14} \\ &+ 6X^{27}Y^{14} + 8X^{28}Y^{14} + 13X^{29}Y^{14} + 3X^{30}Y^{14} + 2X^{31}Y^{14} - 5X^{27}Y^{15} \\ &- 5X^{28}Y^{15} - 4X^{29}Y^{15} - 3X^{30}Y^{15} + 9X^{31}Y^{15} + 6X^{32}Y^{15} + 4X^{33}Y^{15} \\ &- 2X^{29}Y^{16} - 3X^{30}Y^{16} - 8X^{31}Y^{16} - 5X^{32}Y^{16} + 6X^{33}Y^{16} + 2X^{34}Y^{16} \\ &+ 6X^{35}Y^{16} + 2X^{36}Y^{16} - 2X^{31}Y^{17} - X^{32}Y^{17} - 11X^{33}Y^{17} - 11X^{34}Y^{17} \\ &- X^{36}Y^{17} + 4X^{37}Y^{17} + X^{38}Y^{17} - 12X^{35}Y^{18} - 11X^{36}Y^{18} - 8X^{37}Y^{18} \\ &- 6X^{38}Y^{18} + 6X^{39}Y^{18} + 2X^{40}Y^{18} + 2X^{35}Y^{19} + 6X^{36}Y^{19} - 6X^{37}Y^{19} \\ &- 8X^{38}Y^{19} - 11X^{39}Y^{19} - 12X^{40}Y^{19} + X^{37}Y^{20} + 4X^{38}Y^{20} - X^{39}Y^{20} \\ &- 11X^{41}Y^{20} - 11X^{42}Y^{20} - X^{43}Y^{20} - 2X^{44}Y^{20} + 2X^{39}Y^{21} + 6X^{40}Y^{21} \end{aligned}$$

$$\begin{aligned}
& + 2X^{41}Y^{21} + 6X^{42}Y^{21} - 5X^{43}Y^{21} - 8X^{44}Y^{21} - 3X^{45}Y^{21} - 2X^{46}Y^{21} \\
& + 4X^{42}Y^{22} + 6X^{43}Y^{22} + 9X^{44}Y^{22} - 3X^{45}Y^{22} - 4X^{46}Y^{22} - 5X^{47}Y^{22} \\
& - 5X^{48}Y^{22} + 2X^{44}Y^{23} + 3X^{45}Y^{23} + 13X^{46}Y^{23} + 8X^{47}Y^{23} + 6X^{48}Y^{23} \\
& - 2X^{49}Y^{23} - 3X^{50}Y^{23} + X^{46}Y^{24} + 5X^{48}Y^{24} + 3X^{49}Y^{24} + 8X^{50}Y^{24} \\
& - 2X^{51}Y^{24} - 3X^{52}Y^{24} - X^{48}Y^{25} + X^{49}Y^{25} + 8X^{50}Y^{25} + 8X^{51}Y^{25} \\
& + 13X^{52}Y^{25} + 2X^{53}Y^{25} - 2X^{50}Y^{26} - 4X^{51}Y^{26} - 4X^{52}Y^{26} + X^{53}Y^{26} \\
& + 11X^{54}Y^{26} + X^{55}Y^{26} - X^{57}Y^{26} - X^{52}Y^{27} - 4X^{54}Y^{27} - X^{55}Y^{27} \\
& + 8X^{56}Y^{27} + 3X^{57}Y^{27} + 4X^{58}Y^{27} - X^{54}Y^{28} - 2X^{55}Y^{28} - 7X^{56}Y^{28} \\
& - 2X^{57}Y^{28} + X^{58}Y^{28} - 2X^{59}Y^{28} + 2X^{60}Y^{28} - 6X^{58}Y^{29} - 2X^{59}Y^{29} \\
& - X^{60}Y^{29} + X^{61}Y^{29} + 5X^{62}Y^{29} - 3X^{60}Y^{30} - 2X^{61}Y^{30} - 8X^{62}Y^{30} \\
& - 5X^{63}Y^{30} + X^{64}Y^{30} - X^{62}Y^{31} + 3X^{63}Y^{31} - 3X^{64}Y^{31} - X^{65}Y^{31} \\
& + X^{66}Y^{31} + X^{65}Y^{32} - 5X^{66}Y^{32} - 3X^{67}Y^{32} + X^{66}Y^{33} + 4X^{67}Y^{33} \\
& - X^{68}Y^{33} - X^{69}Y^{33} + 2X^{69}Y^{34} - X^{70}Y^{34} + 2X^{71}Y^{35} + X^{73}Y^{36} + X^{75}Y^{37}.
\end{aligned}$$

$\zeta_{\mathfrak{g}_{6,12}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{\mathfrak{g}_{6,12,p}}(s) \Big|_{p \rightarrow p^{-1}} = p^{15-6s} \zeta_{\mathfrak{g}_{6,12,p}}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{\mathfrak{g}_{6,12}}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}
& \zeta_p(s) \zeta_p(s-1) \zeta_p(s-3) \zeta_p(2s-5) \zeta_p(3s-5) \zeta_p(3s-6) \zeta_p(4s-8) \zeta_p(4s-9) \\
& \times \zeta_p(5s-12) \zeta_p(6s-12) \zeta_p(6s-13) \zeta_p(7s-16) W_1(p, p^{-s}) W_2(p, p^{-s}) \\
& \times W_3(p, p^{-s}) W_4(p, p^{-s}) W_5(p, p^{-s})
\end{aligned}$$

where

$$\begin{aligned}W_1(X, Y) &= 1 - X^{21}Y^9, \\W_2(X, Y) &= -1 + 2X^{15}Y^7, \\W_3(X, Y) &= 2 - X^{21}Y^{10}, \\W_4(X, Y) &= -1 + X^9Y^5, \\W_5(X, Y) &= 1 - X^3Y^2 + X^9Y^6.\end{aligned}$$

The ghost is unfriendly.

6 Natural boundary

$\zeta_{\mathfrak{g}_{6,12}}(s)$ has a natural boundary at $\Re(s) = 7/3$, and is of type III.