The zeta function of $\mathfrak{g}_{6,10}(\gamma)$ counting ideals

1 Presentation

For γ squarefree, $\mathfrak{g}_{6,10}(\gamma)$ has presentation

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6 \;\middle|\; \begin{array}{c} [x_1, x_2] = x_4, [x_1, x_4] = x_6, [x_1, x_3] = x_5, \\ [x_2, x_3] = x_6, [x_2, x_4] = \alpha x_5 + \beta x_6 \end{array} \right\rangle.$$

where

$$\alpha x_5 + \beta x_6 = \begin{cases} \gamma x_5 & \text{if } \gamma \equiv 2, 3(\pmod 4) \\ \frac{1}{4}(\gamma - 1)x_5 + x_6 & \text{if } \gamma \equiv 1(\pmod 4) \end{cases}$$

$\mathfrak{g}_{6,10}(\gamma)$ has nilpotency class 3.

$\mathfrak{g}_{6,10}(\gamma)$ can also be constructed for γ not squarefree. If γ' is the squarefree part of γ (i.e. the product of all the distinct primes dividing γ), then for all $p \nmid \gamma$, $\zeta^{\triangleleft}_{\mathfrak{g}_{6,10}(\gamma),p}(s) = \zeta^{\triangleleft}_{\mathfrak{g}_{6,10}(\gamma'),p}(s)$.

2 The local zeta function

The local zeta functions for all but finitely many primes were first calculated by Luke Woodward. The local zeta functions depend on the behaviour of the prime p in the ring of integers of $\mathbb{Q}(\sqrt{\gamma})$. The case where p ramifies has not been calculated. For inert primes p, it is

$$\begin{aligned} \zeta^{\triangleleft}_{\mathfrak{g}_{6,10}(\gamma),p}(s) &= \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(5s-4)\zeta_p(5s-5) \\ &\times \zeta_p(6s-6)\zeta_p(8s-8)\zeta_p(8s-6)^{-1}\zeta_p(10s-8)^{-1}. \end{aligned}$$

For split primes p where $\gamma \not\equiv 1(\pmod 4)$ or $p \nmid \frac{1}{4}(\gamma - 1)$, it is

$$\begin{aligned} \zeta^{\triangleleft}_{\mathfrak{g}_{6,10}(\gamma),p}(s) &= \zeta_p(s)\zeta_p(s-1)^2\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(4s-3)\zeta_p(5s-5)\zeta_p(6s-6) \\ &\times \zeta_p(7s-7)\zeta_p(8s-8)\zeta_p(2s-2)^{-1}W(p, p^{-s}) \end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned} &1 - XY + X^2Y^2 - X^3Y^3 + X^3Y^4 + X^4Y^4 - 2X^3Y^5 - X^5Y^5 + 2X^4Y^6 \\ &+ X^6Y^6 - 2X^5Y^7 - 2X^6Y^7 + 3X^6Y^8 - 4X^7Y^9 + 4X^8Y^{10} - 4X^9Y^{11} \\ &- X^{10}Y^{11} + X^9Y^{12} + 4X^{10}Y^{12} - 4X^{11}Y^{13} + 4X^{12}Y^{14} - 3X^{13}Y^{15} \\ &+ 2X^{13}Y^{16} + 2X^{14}Y^{16} - X^{13}Y^{17} - 2X^{15}Y^{17} + X^{14}Y^{18} + 2X^{16}Y^{18} \\ &- X^{15}Y^{19} - X^{16}Y^{19} + X^{16}Y^{20} - X^{17}Y^{21} + X^{18}Y^{22} - X^{19}Y^{23}. \end{aligned}$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{6,10}(\gamma)}(s)$ is finitely uniform.

3 Functional equation

For p split or inert, the local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,10}(\gamma),p}(s)|_{p\to p^{-1}} = p^{15-12s}\zeta^{\triangleleft}_{\mathfrak{g}_{6,10}(\gamma),p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence and order of pole are unknown since the local zeta functions at ramified primes have not been calculated.

5 Ghost zeta function

The ghost zeta function is unknown, since the local zeta functions vary with the prime.

6 Natural boundary

The natural boundary of $\zeta^{\triangleleft}_{\mathfrak{g}_{6,10}(\gamma)}(s)$ is unknown.

7 Notes

When p splits, $\mathfrak{g}_{6,10}(\gamma) \cong \mathfrak{g}_{6,8}$ as $\mathbb{Z}_p$-Lie rings, so that $\zeta^{\triangleleft}_{\mathfrak{g}_{6,8},p}(s)$. This was indeed how the above zeta function was calculated.