The zeta function of T_4 counting ideals

1 Presentation

T_4 has presentation

$$\langle x_1, x_2, x_3, x_4, y_1, y_2, y_3 \mid [x_1, x_2] = y_1, [x_2, x_3] = y_2, [x_3, x_4] = y_3 \rangle .$$

T_4 has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Gareth Taylor. It is

$$\begin{aligned}\zeta^{\triangleleft}_{T_4,p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-5)^2\zeta_p(5s-6)\zeta_p(5s-8) \\ & \times \zeta_p(6s-10)\zeta_p(7s-12)W(p,p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned}&1 + X^4Y^3 - X^5Y^5 + X^8Y^5 - X^8Y^6 - X^9Y^6 - X^{10}Y^8 - X^{12}Y^8 - X^{13}Y^9 \\ &+ X^{13}Y^{10} - 2X^{14}Y^{10} + X^{14}Y^{11} + X^{15}Y^{11} - X^{16}Y^{11} - X^{17}Y^{11} + 2X^{17}Y^{12} \\ &- X^{18}Y^{12} + X^{18}Y^{13} + X^{19}Y^{14} + X^{21}Y^{14} + X^{22}Y^{16} + X^{23}Y^{16} - X^{23}Y^{17} \\ &+ X^{26}Y^{17} - X^{27}Y^{19} - X^{31}Y^{22}.\end{aligned}$$

$\zeta^{\triangleleft}_{T_4}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{T_4,p}(s)\Big|_{p\to p^{-1}} = -p^{21-11s}\zeta^{\triangleleft}_{T_4,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{T_4}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-5)^2\zeta_p(5s-6)\zeta_p(5s-8)\zeta_p(6s-10)$$
$$\times \zeta_p(7s-12)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})$$

where

$$W_1(X,Y) = 1 + X^8Y^5,$$
$$W_2(X,Y) = 1 - X^9Y^6 + X^{18}Y^{12},$$
$$W_3(X,Y) = 1 - X^5Y^5.$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{T_4}(s)$ has a natural boundary at $\Re(s) = 8/5$, and is of type III.

7 Notes

This ideal zeta function is identical to those of $\mathfrak{g}_{37C}$, though the Lie rings themselves are non-isomorphic.