The zeta function of $Q_5 \times \mathbb{Z}$ counting all subrings

1 Presentation

$Q_5 \times \mathbb{Z}$ has presentation

$$\langle x_1, x_2, x_3, a, x_4, x_5 \mid [x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5 \rangle .$$

$Q_5 \times \mathbb{Z}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta_{Q_5\times\mathbb{Z},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-5)\zeta_p(3s-5)\zeta_p(3s-7) \\ & \times \zeta_p(4s-9)\zeta_p(4s-10)\zeta_p(5s-12)\zeta_p(6s-13)\zeta_p(6s-14) \\ & \times W(p, p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned}
& 1 + X^4Y^2 - X^5Y^3 + X^6Y^3 - X^6Y^4 - X^7Y^4 + X^8Y^4 - 2X^9Y^5 - 2X^{10}Y^5 \\
& - 2X^{11}Y^5 + X^{10}Y^6 + X^{11}Y^6 + X^{12}Y^6 - X^{13}Y^6 - X^{14}Y^6 - 2X^{13}Y^7 \\
& - 2X^{14}Y^7 - 2X^{15}Y^7 + 2X^{14}Y^8 + 2X^{15}Y^8 + 2X^{16}Y^8 - 2X^{17}Y^8 - 4X^{18}Y^8 \\
& - X^{19}Y^8 + X^{17}Y^9 + 4X^{18}Y^9 + 4X^{19}Y^9 + 2X^{20}Y^9 + X^{20}Y^{10} - X^{21}Y^{10} \\
& + X^{21}Y^{11} + 2X^{22}Y^{11} + 6X^{23}Y^{11} + 5X^{24}Y^{11} + X^{25}Y^{11} - X^{26}Y^{11} \\
& - X^{22}Y^{12} - X^{23}Y^{12} - 3X^{24}Y^{12} - X^{25}Y^{12} + 3X^{26}Y^{12} + 4X^{27}Y^{12} \\
& + 2X^{28}Y^{12} - 2X^{26}Y^{13} + X^{28}Y^{13} + X^{30}Y^{13} + X^{31}Y^{13} - X^{26}Y^{14} \\
& - 4X^{28}Y^{14} - 3X^{29}Y^{14} - 2X^{30}Y^{14} + 2X^{31}Y^{14} - 3X^{30}Y^{15} - 2X^{31}Y^{15} \\
& - 4X^{32}Y^{15} - 2X^{33}Y^{15} + X^{35}Y^{15} + X^{31}Y^{16} - 2X^{33}Y^{16} - 4X^{34}Y^{16} \\
& - 2X^{35}Y^{16} - 3X^{36}Y^{16} + 2X^{35}Y^{17} - 2X^{36}Y^{17} - 3X^{37}Y^{17} - 4X^{38}Y^{17} \\
& - X^{40}Y^{17} + X^{35}Y^{18} + X^{36}Y^{18} + X^{38}Y^{18} - 2X^{40}Y^{18} + 2X^{38}Y^{19} \\
& + 4X^{39}Y^{19} + 3X^{40}Y^{19} - X^{41}Y^{19} - 3X^{42}Y^{19} - X^{43}Y^{19} - X^{44}Y^{19} \\
& - X^{40}Y^{20} + X^{41}Y^{20} + 5X^{42}Y^{20} + 6X^{43}Y^{20} + 2X^{44}Y^{20} + X^{45}Y^{20} \\
& - X^{45}Y^{21} + X^{46}Y^{21} + 2X^{46}Y^{22} + 4X^{47}Y^{22} + 4X^{48}Y^{22} + X^{49}Y^{22} \\
& - X^{47}Y^{23} - 4X^{48}Y^{23} - 2X^{49}Y^{23} + 2X^{50}Y^{23} + 2X^{51}Y^{23} + 2X^{52}Y^{23} \\
& - 2X^{51}Y^{24} - 2X^{52}Y^{24} - 2X^{53}Y^{24} - X^{52}Y^{25} - X^{53}Y^{25} + X^{54}Y^{25} \\
& + X^{55}Y^{25} + X^{56}Y^{25} - 2X^{55}Y^{26} - 2X^{56}Y^{26} - 2X^{57}Y^{26} + X^{58}Y^{27}
\end{aligned}$$

$$- X^{59}Y^{27} - X^{60}Y^{27} + X^{60}Y^{28} - X^{61}Y^{28} + X^{62}Y^{29} + X^{66}Y^{31}.$$

$\zeta_{Q_5\times\mathbb{Z}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{Q_5\times\mathbb{Z},p}(s)\big|_{p\to p^{-1}} = p^{15-6s}\zeta_{Q_5\times\mathbb{Z},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{Q_5\times\mathbb{Z}}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}
&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-5)\zeta_p(3s-5)\zeta_p(3s-7)\zeta_p(4s-9)\\
&\times \zeta_p(4s-10)\zeta_p(5s-12)\zeta_p(6s-13)\zeta_p(6s-14)W_1(p,p^{-s})W_2(p,p^{-s})\\
&\times W_3(p,p^{-s})W_4(p,p^{-s})
\end{aligned}$$

where

$$\begin{aligned}
W_1(X,Y) &= 1 + X^{31}Y^{13},\\
W_2(X,Y) &= 1 - X^{9}Y^{4},\\
W_3(X,Y) &= -1 - X^{4}Y^{2} + 2X^{12}Y^{6} + X^{16}Y^{8} - X^{20}Y^{10},\\
W_4(X,Y) &= -1 + X^{6}Y^{4}.
\end{aligned}$$

The ghost is unfriendly.

6 Natural boundary

$\zeta_{Q_5\times\mathbb{Z}}(s)$ has a natural boundary at $\Re(s) = 31/13$, and is of type III.