

# The zeta function of $Q_5 \times \mathbb{Z}$ counting all subrings

## 1 Presentation

$Q_5 \times \mathbb{Z}$  has presentation

$$\langle x_1, x_2, x_3, a, x_4, x_5 \mid [x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5 \rangle.$$

$Q_5 \times \mathbb{Z}$  has nilpotency class 3.

## 2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta_{Q_5 \times \mathbb{Z}, p}(s) &= \zeta_p(s) \zeta_p(s-1) \zeta_p(s-2) \zeta_p(s-3) \zeta_p(2s-5) \zeta_p(3s-5) \zeta_p(3s-7) \\ &\quad \times \zeta_p(4s-9) \zeta_p(4s-10) \zeta_p(5s-12) \zeta_p(6s-13) \zeta_p(6s-14) \\ &\quad \times W(p, p^{-s}) \end{aligned}$$

where  $W(X, Y)$  is

$$\begin{aligned} &1 + X^4 Y^2 - X^5 Y^3 + X^6 Y^3 - X^6 Y^4 - X^7 Y^4 + X^8 Y^4 - 2X^9 Y^5 - 2X^{10} Y^5 \\ &- 2X^{11} Y^5 + X^{10} Y^6 + X^{11} Y^6 + X^{12} Y^6 - X^{13} Y^6 - X^{14} Y^6 - 2X^{13} Y^7 \\ &- 2X^{14} Y^7 - 2X^{15} Y^7 + 2X^{14} Y^8 + 2X^{15} Y^8 + 2X^{16} Y^8 - 2X^{17} Y^8 - 4X^{18} Y^8 \\ &- X^{19} Y^8 + X^{17} Y^9 + 4X^{18} Y^9 + 4X^{19} Y^9 + 2X^{20} Y^9 + X^{20} Y^{10} - X^{21} Y^{10} \\ &+ X^{21} Y^{11} + 2X^{22} Y^{11} + 6X^{23} Y^{11} + 5X^{24} Y^{11} + X^{25} Y^{11} - X^{26} Y^{11} \\ &- X^{22} Y^{12} - X^{23} Y^{12} - 3X^{24} Y^{12} - X^{25} Y^{12} + 3X^{26} Y^{12} + 4X^{27} Y^{12} \\ &+ 2X^{28} Y^{12} - 2X^{26} Y^{13} + X^{28} Y^{13} + X^{30} Y^{13} + X^{31} Y^{13} - X^{26} Y^{14} \\ &- 4X^{28} Y^{14} - 3X^{29} Y^{14} - 2X^{30} Y^{14} + 2X^{31} Y^{14} - 3X^{30} Y^{15} - 2X^{31} Y^{15} \\ &- 4X^{32} Y^{15} - 2X^{33} Y^{15} + X^{35} Y^{15} + X^{31} Y^{16} - 2X^{33} Y^{16} - 4X^{34} Y^{16} \\ &- 2X^{35} Y^{16} - 3X^{36} Y^{16} + 2X^{35} Y^{17} - 2X^{36} Y^{17} - 3X^{37} Y^{17} - 4X^{38} Y^{17} \\ &- X^{40} Y^{17} + X^{35} Y^{18} + X^{36} Y^{18} + X^{38} Y^{18} - 2X^{40} Y^{18} + 2X^{38} Y^{19} \\ &+ 4X^{39} Y^{19} + 3X^{40} Y^{19} - X^{41} Y^{19} - 3X^{42} Y^{19} - X^{43} Y^{19} - X^{44} Y^{19} \\ &- X^{40} Y^{20} + X^{41} Y^{20} + 5X^{42} Y^{20} + 6X^{43} Y^{20} + 2X^{44} Y^{20} + X^{45} Y^{20} \\ &- X^{45} Y^{21} + X^{46} Y^{21} + 2X^{46} Y^{22} + 4X^{47} Y^{22} + 4X^{48} Y^{22} + X^{49} Y^{22} \\ &- X^{47} Y^{23} - 4X^{48} Y^{23} - 2X^{49} Y^{23} + 2X^{50} Y^{23} + 2X^{51} Y^{23} + 2X^{52} Y^{23} \\ &- 2X^{51} Y^{24} - 2X^{52} Y^{24} - 2X^{53} Y^{24} - X^{52} Y^{25} - X^{53} Y^{25} + X^{54} Y^{25} \\ &+ X^{55} Y^{25} + X^{56} Y^{25} - 2X^{55} Y^{26} - 2X^{56} Y^{26} - 2X^{57} Y^{26} + X^{58} Y^{27} \end{aligned}$$

$$-X^{59}Y^{27} - X^{60}Y^{27} + X^{60}Y^{28} - X^{61}Y^{28} + X^{62}Y^{29} + X^{66}Y^{31}.$$

$\zeta_{Q_5 \times \mathbb{Z}}(s)$  is uniform.

### 3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{Q_5 \times \mathbb{Z}, p}(s)|_{p \rightarrow p^{-1}} = p^{15-6s} \zeta_{Q_5 \times \mathbb{Z}, p}(s).$$

### 4 Abscissa of convergence and order of pole

The abscissa of convergence of  $\zeta_{Q_5 \times \mathbb{Z}}(s)$  is 4, with a simple pole at  $s = 4$ .

### 5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned} & \zeta_p(s) \zeta_p(s-1) \zeta_p(s-2) \zeta_p(s-3) \zeta_p(2s-5) \zeta_p(3s-5) \zeta_p(3s-7) \zeta_p(4s-9) \\ & \times \zeta_p(4s-10) \zeta_p(5s-12) \zeta_p(6s-13) \zeta_p(6s-14) W_1(p, p^{-s}) W_2(p, p^{-s}) \\ & \times W_3(p, p^{-s}) W_4(p, p^{-s}) \end{aligned}$$

where

$$\begin{aligned} W_1(X, Y) &= 1 + X^{31}Y^{13}, \\ W_2(X, Y) &= 1 - X^9Y^4, \\ W_3(X, Y) &= -1 - X^4Y^2 + 2X^{12}Y^6 + X^{16}Y^8 - X^{20}Y^{10}, \\ W_4(X, Y) &= -1 + X^6Y^4. \end{aligned}$$

The ghost is unfriendly.

### 6 Natural boundary

$\zeta_{Q_5 \times \mathbb{Z}}(s)$  has a natural boundary at  $\Re(s) = 31/13$ , and is of type III.