The zeta function of M_4 counting ideals

1 Presentation

M_4 has presentation

$$\langle z, x_1, x_2, x_3, x_4 \mid [z, x_1] = x_2, [z, x_2] = x_3, [z, x_3] = x_4 \rangle .$$

M_4 has nilpotency class 4.

2 The local zeta function

The local zeta function was first calculated by Gareth Taylor. It is

$$\begin{aligned}\zeta^{\triangleleft}_{M_4,p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(3s-2)\zeta_p(5s-2)\zeta_p(7s-4)\zeta_p(8s-5)\zeta_p(9s-6) \\ & \times \zeta_p(11s-6)\zeta_p(12s-7)\zeta_p(6s-3)^{-1}W(p,p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned}&1 + X^2Y^4 - X^2Y^5 + X^3Y^5 - X^2Y^6 + 2X^3Y^6 - X^3Y^7 - X^5Y^9 + X^6Y^{10} \\ &- 2X^5Y^{11} - X^7Y^{13} - X^8Y^{13} + X^7Y^{14} - X^8Y^{14} - X^8Y^{15} - X^9Y^{15} \\ &+ X^9Y^{16} - X^9Y^{17} - X^{10}Y^{17} + 2X^9Y^{18} - X^{10}Y^{18} + X^{10}Y^{19} - 2X^{11}Y^{19} \\ &+ X^{10}Y^{20} + X^{11}Y^{20} - X^{11}Y^{21} + X^{11}Y^{22} + X^{12}Y^{22} + X^{12}Y^{23} - X^{13}Y^{23} \\ &+ X^{12}Y^{24} + X^{13}Y^{24} + 2X^{15}Y^{26} - X^{14}Y^{27} + X^{15}Y^{28} + X^{17}Y^{30} - 2X^{17}Y^{31} \\ &+ X^{18}Y^{31} - X^{17}Y^{32} + X^{18}Y^{32} - X^{18}Y^{33} - X^{20}Y^{37}.\end{aligned}$$

$\zeta^{\triangleleft}_{M_4}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{M_4,p}(s)\big|_{p\to p^{-1}} = -p^{10-14s}\zeta^{\triangleleft}_{M_4,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{M_4}(s)$ is 2, with a simple pole at $s = 2$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(3s-2)\zeta_p(5s-2)\zeta_p(7s-4)\zeta_p(8s-5)\zeta_p(9s-6)\zeta_p(11s-6)$$
$$\times \zeta_p(12s-7)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})W_4(p,p^{-s})$$

where

$$\begin{aligned}
W_1(X,Y) &= 1 - X^8Y^{13}, \\
W_2(X,Y) &= -1 + X^{10}Y^{18}, \\
W_3(X,Y) &= 1 - X^3Y^6, \\
W_4(X,Y) &= -1 + X^2Y^6.
\end{aligned}$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{M_4}(s)$ has a natural boundary at $\Re(s) = 8/13$, and is of type III.