The zeta function of $M_3 \times \mathbb{Z}$ counting all subrings

1 Presentation

$M_3 \times \mathbb{Z}$ has presentation

$$\langle z, x_1, x_2, a, x_3 \mid [z, x_1] = x_2, [z, x_2] = x_3 \rangle .$$

$M_3 \times \mathbb{Z}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta_{M_3 \times \mathbb{Z},p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)^2\zeta_p(2s-4)\zeta_p(3s-6)\zeta_p(4s-8)W(p,p^{-s})$$

where $W(X,Y)$ is

$$1 - X^2Y + X^3Y^2 + X^4Y^2 - X^4Y^3 - X^5Y^3 - X^8Y^5 - X^9Y^5 + X^9Y^6 + X^{10}Y^6 - X^{11}Y^7 + X^{13}Y^8.$$

$\zeta_{M_3 \times \mathbb{Z}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{M_3 \times \mathbb{Z},p}(s)\big|_{p \to p^{-1}} = -p^{10-5s}\zeta_{M_3 \times \mathbb{Z},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{M_3 \times \mathbb{Z}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)^2\zeta_p(2s-4)\zeta_p(3s-6)\zeta_p(4s-8)W_1(p,p^{-s})W_2(p,p^{-s}) \times W_3(p,p^{-s})$$

where

$$W_1(X,Y) = 1 - X^2Y + X^4Y^2,$$
$$W_2(X,Y) = 1 - X^5Y^3,$$
$$W_3(X,Y) = -1 + X^4Y^3.$$

The ghost is friendly.

6 Natural boundary

$\zeta_{M_3\times\mathbb{Z}}(s)$ has a natural boundary at $\Re(s) = 2$, and is of type II.