The zeta function of $M_3 \times \mathbb{Z}^2$ counting all subrings

1 Presentation

$M_3 \times \mathbb{Z}^2$ has presentation

$$\langle z, x_1, x_2, a_1, a_2, x_3 \mid [z, x_1] = x_2, [z, x_2] = x_3 \rangle .$$

$M_3 \times \mathbb{Z}^2$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta_{M_3 \times \mathbb{Z}^2,p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-5)\zeta_p(3s-7)\zeta_p(3s-8) \\ & \times \zeta_p(4s-10)W(p, p^{-s})\end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned}&1 + X^4Y^2 + X^5Y^2 - X^5Y^3 - X^7Y^4 + X^{10}Y^4 - 2X^{10}Y^5 - 2X^{11}Y^5 + X^{11}Y^6 \\ &- X^{14}Y^6 - X^{16}Y^7 + X^{16}Y^8 + X^{17}Y^8 + X^{21}Y^{10}.\end{aligned}$$

$\zeta_{M_3 \times \mathbb{Z}^2}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\left.\zeta_{M_3 \times \mathbb{Z}^2,p}(s)\right|_{p \to p^{-1}} = p^{15-6s}\zeta_{M_3 \times \mathbb{Z}^2,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{M_3 \times \mathbb{Z}^2}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-5)\zeta_p(3s-7)\zeta_p(3s-8)\zeta_p(4s-10) \\ &\times W_1(p, p^{-s})W_2(p, p^{-s})W_3(p, p^{-s})\end{aligned}$$

where

$$
\begin{aligned}
W_1(X,Y) &= 1 + X^5Y^2 + X^{10}Y^4, \\
W_2(X,Y) &= 1 - X^4Y^2 - X^6Y^3, \\
W_3(X,Y) &= -1 + X^5Y^3.
\end{aligned}
$$

The ghost is unfriendly.

6 Natural boundary

$\zeta_{M_3\times\mathbb{Z}^2}(s)$ has a natural boundary at $\Re(s) = 5/2$, and is of type II.