The zeta function of $M_3 \times M_3$ counting ideals

1 Presentation

$M_3 \times M_3$ has presentation

$$\left\langle z_1, z_2, w_1, w_2, x_1, x_2, y_1, y_2 \;\middle|\; \begin{array}{l} [z_1, w_1] = x_1, [z_1, x_1] = y_1, \\ [z_2, w_2] = x_2, [z_2, x_2] = y_2 \end{array} \right\rangle.$$

$M_3 \times M_3$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta^{\triangleleft}_{M_3 \times M_3, p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-2)\zeta_p(3s-4)^2\zeta_p(4s-4) \\ & \times \zeta_p(5s-5)\zeta_p(6s-5)\zeta_p(7s-5)\zeta_p(7s-6)\zeta_p(8s-6) \\ & \times \zeta_p(9s-7)\zeta_p(9s-10)\zeta_p(10s-10)\zeta_p(11s-11)\zeta_p(12s-12) \\ & \times \zeta_p(13s-15)W(p, p^{-s}) \end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned} & 1 - X^2Y^2 + X^4Y^4 - 3X^4Y^5 + 2X^5Y^5 - 2X^4Y^6 + X^5Y^6 - X^6Y^6 + X^4Y^7 \\ & - 4X^5Y^7 + 4X^6Y^7 - 2X^7Y^7 + 2X^4Y^8 - 4X^5Y^8 + 3X^6Y^8 - X^7Y^8 + X^8Y^8 \\ & - X^4Y^9 + 3X^5Y^9 - 4X^6Y^9 + 4X^7Y^9 - 2X^8Y^9 - 2X^9Y^9 + X^{10}Y^9 \\ & - 2X^6Y^{10} + 4X^7Y^{10} - 5X^9Y^{10} + X^{10}Y^{10} + X^6Y^{11} - 3X^7Y^{11} + 2X^8Y^{11} \\ & + 7X^9Y^{11} - 8X^{10}Y^{11} + 3X^{11}Y^{11} - X^{12}Y^{11} + 2X^8Y^{12} + 3X^9Y^{12} \\ & - 5X^{10}Y^{12} + 4X^{11}Y^{12} - X^{12}Y^{12} - X^8Y^{13} + 6X^9Y^{13} + X^{10}Y^{13} \\ & - 10X^{11}Y^{13} + 7X^{12}Y^{13} - 2X^{14}Y^{13} - 4X^9Y^{14} + 11X^{10}Y^{14} - 9X^{11}Y^{14} \\ & + 5X^{12}Y^{14} - 2X^{13}Y^{14} - X^9Y^{15} + X^{10}Y^{15} + X^{11}Y^{15} - 4X^{12}Y^{15} \\ & + 11X^{13}Y^{15} - 4X^{14}Y^{15} - 2X^{15}Y^{15} + 2X^{16}Y^{15} - X^{10}Y^{16} + 6X^{11}Y^{16} \\ & - 10X^{12}Y^{16} - 2X^{13}Y^{16} + 22X^{14}Y^{16} - 11X^{15}Y^{16} + X^{16}Y^{16} - X^{10}Y^{17} \\ & + 2X^{11}Y^{17} - 6X^{13}Y^{17} + 7X^{14}Y^{17} + 4X^{15}Y^{17} - 4X^{16}Y^{17} + 2X^{17}Y^{17} \\ & - 2X^{18}Y^{17} + X^{12}Y^{18} - 6X^{13}Y^{18} - 9X^{14}Y^{18} + 27X^{15}Y^{18} - 24X^{16}Y^{18} \\ & + 8X^{17}Y^{18} + 5X^{18}Y^{18} - 3X^{19}Y^{18} + X^{12}Y^{19} + 2X^{13}Y^{19} - 19X^{14}Y^{19} \\ & + 17X^{15}Y^{19} + X^{16}Y^{19} - 7X^{17}Y^{19} + 2X^{18}Y^{19} + 7X^{19}Y^{19} - 2X^{20}Y^{19} \\ & + 5X^{14}Y^{20} - 25X^{15}Y^{20} + 34X^{16}Y^{20} - 25X^{17}Y^{20} + 7X^{18}Y^{20} + 7X^{19}Y^{20} \end{aligned}$$

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\begin{aligned}
&- 4X^{20}Y^{20} + 2X^{21}Y^{20} - 9X^{15}Y^{21} + 16X^{16}Y^{21} - 7X^{17}Y^{21} - 21X^{18}Y^{21} \\
&+ 19X^{19}Y^{21} + 3X^{20}Y^{21} - 8X^{21}Y^{21} + 2X^{22}Y^{21} + 2X^{14}Y^{22} - X^{15}Y^{22} \\
&- 16X^{16}Y^{22} + 28X^{17}Y^{22} - 14X^{18}Y^{22} - 33X^{19}Y^{22} + 37X^{20}Y^{22} - 14X^{21}Y^{22} \\
&+ 3X^{22}Y^{22} - X^{23}Y^{22} + X^{24}Y^{22} + 3X^{15}Y^{23} - 7X^{16}Y^{23} + 11X^{17}Y^{23} \\
&- 3X^{18}Y^{23} - 21X^{19}Y^{23} + 9X^{20}Y^{23} + 3X^{21}Y^{23} - 8X^{22}Y^{23} + 11X^{23}Y^{23} \\
&- 2X^{24}Y^{23} + X^{16}Y^{24} - 5X^{17}Y^{24} + 21X^{18}Y^{24} - 15X^{19}Y^{24} - 40X^{20}Y^{24} \\
&+ 59X^{21}Y^{24} - 29X^{22}Y^{24} - 12X^{23}Y^{24} + 16X^{24}Y^{24} - X^{25}Y^{24} - X^{26}Y^{24} \\
&- 3X^{17}Y^{25} + 3X^{18}Y^{25} + 27X^{19}Y^{25} - 51X^{20}Y^{25} + 23X^{21}Y^{25} + 2X^{22}Y^{25} \\
&- 8X^{23}Y^{25} - 5X^{24}Y^{25} + 3X^{25}Y^{25} - 3X^{18}Y^{26} + 8X^{19}Y^{26} + 17X^{20}Y^{26} \\
&- 51X^{21}Y^{26} + 57X^{22}Y^{26} - 36X^{23}Y^{26} - 17X^{24}Y^{26} + 28X^{25}Y^{26} - 14X^{26}Y^{26} \\
&+ X^{27}Y^{26} + X^{28}Y^{26} - 4X^{19}Y^{27} + 28X^{20}Y^{27} - 45X^{21}Y^{27} + 32X^{22}Y^{27} \\
&+ 24X^{23}Y^{27} - 54X^{24}Y^{27} + 8X^{25}Y^{27} + 14X^{26}Y^{27} - 4X^{27}Y^{27} - 2X^{28}Y^{27} \\
&+ X^{29}Y^{27} + 13X^{21}Y^{28} - 35X^{22}Y^{28} + 44X^{23}Y^{28} + 23X^{24}Y^{28} - 80X^{25}Y^{28} \\
&+ 53X^{26}Y^{28} - 26X^{27}Y^{28} + 7X^{28}Y^{28} + 2X^{29}Y^{28} - X^{30}Y^{28} - 3X^{20}Y^{29} \\
&+ 16X^{21}Y^{29} - 30X^{22}Y^{29} + 19X^{23}Y^{29} + 37X^{24}Y^{29} - 48X^{25}Y^{29} + 18X^{26}Y^{29} \\
&+ 12X^{27}Y^{29} - 28X^{28}Y^{29} + 4X^{29}Y^{29} + 4X^{30}Y^{29} - X^{31}Y^{29} + X^{21}Y^{30} \\
&+ 2X^{22}Y^{30} - 17X^{23}Y^{30} + 25X^{24}Y^{30} + 44X^{25}Y^{30} - 101X^{26}Y^{30} + 81X^{27}Y^{30} \\
&- 7X^{28}Y^{30} - 38X^{29}Y^{30} + 14X^{30}Y^{30} - 2X^{31}Y^{30} + X^{32}Y^{30} - 2X^{21}Y^{31} \\
&+ 7X^{22}Y^{31} - 15X^{23}Y^{31} - 6X^{24}Y^{31} + 69X^{25}Y^{31} - 67X^{26}Y^{31} + 23X^{27}Y^{31} \\
&+ 17X^{28}Y^{31} - 6X^{29}Y^{31} - 14X^{30}Y^{31} + 9X^{31}Y^{31} - 4X^{32}Y^{31} + X^{33}Y^{31} \\
&- X^{23}Y^{32} - 11X^{24}Y^{32} + 9X^{25}Y^{32} + 48X^{26}Y^{32} - 90X^{27}Y^{32} + 82X^{28}Y^{32} \\
&+ 5X^{29}Y^{32} - 58X^{30}Y^{32} + 42X^{31}Y^{32} - 9X^{32}Y^{32} - 5X^{33}Y^{32} + X^{34}Y^{32} \\
&+ 2X^{23}Y^{33} - 3X^{24}Y^{33} - 26X^{25}Y^{33} + 68X^{26}Y^{33} - 71X^{27}Y^{33} + 15X^{28}Y^{33} \\
&+ 67X^{29}Y^{33} - 32X^{30}Y^{33} - 22X^{31}Y^{33} + 17X^{32}Y^{33} + 2X^{33}Y^{33} - 6X^{34}Y^{33} \\
&- 3X^{25}Y^{34} - 4X^{26}Y^{34} + 32X^{27}Y^{34} - 69X^{28}Y^{34} + 38X^{29}Y^{34} + 73X^{30}Y^{34} \\
&- 95X^{31}Y^{34} + 66X^{32}Y^{34} - 28X^{33}Y^{34} + X^{34}Y^{34} + 2X^{35}Y^{34} - X^{36}Y^{34} \\
&+ X^{25}Y^{35} - 15X^{26}Y^{35} + 39X^{27}Y^{35} - 50X^{28}Y^{35} - 25X^{29}Y^{35} + 100X^{30}Y^{35} \\
&- 55X^{31}Y^{35} - 8X^{32}Y^{35} + 51X^{33}Y^{35} - 21X^{34}Y^{35} - 10X^{35}Y^{35} + 7X^{36}Y^{35} \\
&- X^{26}Y^{36} - 2X^{27}Y^{36} + 14X^{28}Y^{36} - 44X^{29}Y^{36} - 5X^{30}Y^{36} + 105X^{31}Y^{36} \\
&- 126X^{32}Y^{36} + 69X^{33}Y^{36} + 24X^{34}Y^{36} - 29X^{35}Y^{36} + 6X^{36}Y^{36} - 2X^{37}Y^{36} \\
&+ X^{38}Y^{36} - 3X^{27}Y^{37} + 18X^{28}Y^{37} - 21X^{29}Y^{37} - 52X^{30}Y^{37} + 100X^{31}Y^{37} \\
&- 67X^{32}Y^{37} - 14X^{33}Y^{37} + 60X^{34}Y^{37} - 11X^{35}Y^{37} - 15X^{36}Y^{37} + 15X^{37}Y^{37} \\
&- 7X^{38}Y^{37} + X^{28}Y^{38} + 17X^{29}Y^{38} - 42X^{30}Y^{38} - 8X^{31}Y^{38} + 84X^{32}Y^{38} \\
&- 123X^{33}Y^{38} + 37X^{34}Y^{38} + 82X^{35}Y^{38} - 75X^{36}Y^{38} + 25X^{37}Y^{38} + 5X^{38}Y^{38}
\end{aligned}
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\begin{aligned}
&- 4X^{39}Y^{38} + X^{29}Y^{39} + 13X^{30}Y^{39} - 61X^{31}Y^{39} + 93X^{32}Y^{39} - 65X^{33}Y^{39} \\
&- 59X^{34}Y^{39} + 83X^{35}Y^{39} - 6X^{36}Y^{39} - 21X^{37}Y^{39} + 16X^{38}Y^{39} + 4X^{39}Y^{39} \\
&- 2X^{40}Y^{39} + 6X^{30}Y^{40} - 16X^{31}Y^{40} - 5X^{32}Y^{40} + 63X^{33}Y^{40} - 98X^{34}Y^{40} \\
&- 17X^{35}Y^{40} + 115X^{36}Y^{40} - 111X^{37}Y^{40} + 55X^{38}Y^{40} + 5X^{39}Y^{40} \\
&- 10X^{40}Y^{40} + 3X^{41}Y^{40} + 9X^{31}Y^{41} - 35X^{32}Y^{41} + 54X^{33}Y^{41} + 2X^{34}Y^{41} \\
&- 120X^{35}Y^{41} + 112X^{36}Y^{41} - 35X^{37}Y^{41} - 44X^{38}Y^{41} + 40X^{39}Y^{41} \\
&+ 12X^{40}Y^{41} - 14X^{41}Y^{41} + 2X^{42}Y^{41} + 3X^{31}Y^{42} - 5X^{32}Y^{42} + 6X^{33}Y^{42} \\
&+ 28X^{34}Y^{42} - 34X^{35}Y^{42} - 64X^{36}Y^{42} + 136X^{37}Y^{42} - 134X^{38}Y^{42} \\
&+ 21X^{39}Y^{42} + 45X^{40}Y^{42} - 25X^{41}Y^{42} + 10X^{42}Y^{42} - 3X^{43}Y^{42} + 2X^{32}Y^{43} \\
&- 12X^{33}Y^{43} + 33X^{34}Y^{43} + 13X^{35}Y^{43} - 108X^{36}Y^{43} + 113X^{37}Y^{43} \\
&- 30X^{38}Y^{43} - 85X^{39}Y^{43} + 58X^{40}Y^{43} + 10X^{41}Y^{43} - 28X^{42}Y^{43} + 18X^{43}Y^{43} \\
&- X^{44}Y^{43} - 3X^{33}Y^{44} - 4X^{34}Y^{44} + 37X^{35}Y^{44} - 31X^{36}Y^{44} - 39X^{37}Y^{44} \\
&+ 112X^{38}Y^{44} - 89X^{39}Y^{44} - 59X^{40}Y^{44} + 104X^{41}Y^{44} - 58X^{42}Y^{44} \\
&+ 6X^{43}Y^{44} + 7X^{44}Y^{44} - 2X^{34}Y^{45} + 2X^{35}Y^{45} + 38X^{36}Y^{45} - 90X^{37}Y^{45} \\
&+ 105X^{38}Y^{45} - 6X^{39}Y^{45} - 106X^{40}Y^{45} + 59X^{41}Y^{45} + 8X^{42}Y^{45} - 34X^{43}Y^{45} \\
&+ 10X^{44}Y^{45} + X^{45}Y^{45} - 4X^{35}Y^{46} + 21X^{36}Y^{46} - 19X^{37}Y^{46} - 24X^{38}Y^{46} \\
&+ 116X^{39}Y^{46} - 70X^{40}Y^{46} - 85X^{41}Y^{46} + 131X^{42}Y^{46} - 88X^{43}Y^{46} \\
&- 2X^{44}Y^{46} + 24X^{45}Y^{46} - 7X^{46}Y^{46} - 6X^{36}Y^{47} + 25X^{37}Y^{47} - 48X^{38}Y^{47} \\
&+ 33X^{39}Y^{47} + 93X^{40}Y^{47} - 156X^{41}Y^{47} + 93X^{42}Y^{47} - 4X^{43}Y^{47} - 52X^{44}Y^{47} \\
&+ 8X^{45}Y^{47} + 14X^{46}Y^{47} - 3X^{47}Y^{47} + 3X^{37}Y^{48} - 14X^{38}Y^{48} - 8X^{39}Y^{48} \\
&+ 52X^{40}Y^{48} + 4X^{41}Y^{48} - 93X^{42}Y^{48} + 156X^{43}Y^{48} - 93X^{44}Y^{48} - 33X^{45}Y^{48} \\
&+ 48X^{46}Y^{48} - 25X^{47}Y^{48} + 6X^{48}Y^{48} + 7X^{38}Y^{49} - 24X^{39}Y^{49} + 2X^{40}Y^{49} \\
&+ 88X^{41}Y^{49} - 131X^{42}Y^{49} + 85X^{43}Y^{49} + 70X^{44}Y^{49} - 116X^{45}Y^{49} \\
&+ 24X^{46}Y^{49} + 19X^{47}Y^{49} - 21X^{48}Y^{49} + 4X^{49}Y^{49} - X^{39}Y^{50} - 10X^{40}Y^{50} \\
&+ 34X^{41}Y^{50} - 8X^{42}Y^{50} - 59X^{43}Y^{50} + 106X^{44}Y^{50} + 6X^{45}Y^{50} - 105X^{46}Y^{50} \\
&+ 90X^{47}Y^{50} - 38X^{48}Y^{50} - 2X^{49}Y^{50} + 2X^{50}Y^{50} - 7X^{40}Y^{51} - 6X^{41}Y^{51} \\
&+ 58X^{42}Y^{51} - 104X^{43}Y^{51} + 59X^{44}Y^{51} + 89X^{45}Y^{51} - 112X^{46}Y^{51} \\
&+ 39X^{47}Y^{51} + 31X^{48}Y^{51} - 37X^{49}Y^{51} + 4X^{50}Y^{51} + 3X^{51}Y^{51} + X^{40}Y^{52} \\
&- 18X^{41}Y^{52} + 28X^{42}Y^{52} - 10X^{43}Y^{52} - 58X^{44}Y^{52} + 85X^{45}Y^{52} + 30X^{46}Y^{52} \\
&- 113X^{47}Y^{52} + 108X^{48}Y^{52} - 13X^{49}Y^{52} - 33X^{50}Y^{52} + 12X^{51}Y^{52} \\
&- 2X^{52}Y^{52} + 3X^{41}Y^{53} - 10X^{42}Y^{53} + 25X^{43}Y^{53} - 45X^{44}Y^{53} - 21X^{45}Y^{53} \\
&+ 134X^{46}Y^{53} - 136X^{47}Y^{53} + 64X^{48}Y^{53} + 34X^{49}Y^{53} - 28X^{50}Y^{53} \\
&- 6X^{51}Y^{53} + 5X^{52}Y^{53} - 3X^{53}Y^{53} - 2X^{42}Y^{54} + 14X^{43}Y^{54} - 12X^{44}Y^{54} \\
&- 40X^{45}Y^{54} + 44X^{46}Y^{54} + 35X^{47}Y^{54} - 112X^{48}Y^{54} + 120X^{49}Y^{54}
\end{aligned}
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\begin{aligned}
&- 2X^{50}Y^{54} - 54X^{51}Y^{54} + 35X^{52}Y^{54} - 9X^{53}Y^{54} - 3X^{43}Y^{55} + 10X^{44}Y^{55} \\
&- 5X^{45}Y^{55} - 55X^{46}Y^{55} + 111X^{47}Y^{55} - 115X^{48}Y^{55} + 17X^{49}Y^{55} \\
&+ 98X^{50}Y^{55} - 63X^{51}Y^{55} + 5X^{52}Y^{55} + 16X^{53}Y^{55} - 6X^{54}Y^{55} + 2X^{44}Y^{56} \\
&- 4X^{45}Y^{56} - 16X^{46}Y^{56} + 21X^{47}Y^{56} + 6X^{48}Y^{56} - 83X^{49}Y^{56} + 59X^{50}Y^{56} \\
&+ 65X^{51}Y^{56} - 93X^{52}Y^{56} + 61X^{53}Y^{56} - 13X^{54}Y^{56} - X^{55}Y^{56} + 4X^{45}Y^{57} \\
&- 5X^{46}Y^{57} - 25X^{47}Y^{57} + 75X^{48}Y^{57} - 82X^{49}Y^{57} - 37X^{50}Y^{57} + 123X^{51}Y^{57} \\
&- 84X^{52}Y^{57} + 8X^{53}Y^{57} + 42X^{54}Y^{57} - 17X^{55}Y^{57} - X^{56}Y^{57} + 7X^{46}Y^{58} \\
&- 15X^{47}Y^{58} + 15X^{48}Y^{58} + 11X^{49}Y^{58} - 60X^{50}Y^{58} + 14X^{51}Y^{58} + 67X^{52}Y^{58} \\
&- 100X^{53}Y^{58} + 52X^{54}Y^{58} + 21X^{55}Y^{58} - 18X^{56}Y^{58} + 3X^{57}Y^{58} - X^{46}Y^{59} \\
&+ 2X^{47}Y^{59} - 6X^{48}Y^{59} + 29X^{49}Y^{59} - 24X^{50}Y^{59} - 69X^{51}Y^{59} + 126X^{52}Y^{59} \\
&- 105X^{53}Y^{59} + 5X^{54}Y^{59} + 44X^{55}Y^{59} - 14X^{56}Y^{59} + 2X^{57}Y^{59} + X^{58}Y^{59} \\
&- 7X^{48}Y^{60} + 10X^{49}Y^{60} + 21X^{50}Y^{60} - 51X^{51}Y^{60} + 8X^{52}Y^{60} + 55X^{53}Y^{60} \\
&- 100X^{54}Y^{60} + 25X^{55}Y^{60} + 50X^{56}Y^{60} - 39X^{57}Y^{60} + 15X^{58}Y^{60} - X^{59}Y^{60} \\
&+ X^{48}Y^{61} - 2X^{49}Y^{61} - X^{50}Y^{61} + 28X^{51}Y^{61} - 66X^{52}Y^{61} + 95X^{53}Y^{61} \\
&- 73X^{54}Y^{61} - 38X^{55}Y^{61} + 69X^{56}Y^{61} - 32X^{57}Y^{61} + 4X^{58}Y^{61} + 3X^{59}Y^{61} \\
&+ 6X^{50}Y^{62} - 2X^{51}Y^{62} - 17X^{52}Y^{62} + 22X^{53}Y^{62} + 32X^{54}Y^{62} - 67X^{55}Y^{62} \\
&- 15X^{56}Y^{62} + 71X^{57}Y^{62} - 68X^{58}Y^{62} + 26X^{59}Y^{62} + 3X^{60}Y^{62} - 2X^{61}Y^{62} \\
&- X^{50}Y^{63} + 5X^{51}Y^{63} + 9X^{52}Y^{63} - 42X^{53}Y^{63} + 58X^{54}Y^{63} - 5X^{55}Y^{63} \\
&- 82X^{56}Y^{63} + 90X^{57}Y^{63} - 48X^{58}Y^{63} - 9X^{59}Y^{63} + 11X^{60}Y^{63} + X^{61}Y^{63} \\
&- X^{51}Y^{64} + 4X^{52}Y^{64} - 9X^{53}Y^{64} + 14X^{54}Y^{64} + 6X^{55}Y^{64} - 17X^{56}Y^{64} \\
&- 23X^{57}Y^{64} + 67X^{58}Y^{64} - 69X^{59}Y^{64} + 6X^{60}Y^{64} + 15X^{61}Y^{64} - 7X^{62}Y^{64} \\
&+ 2X^{63}Y^{64} - X^{52}Y^{65} + 2X^{53}Y^{65} - 14X^{54}Y^{65} + 38X^{55}Y^{65} + 7X^{56}Y^{65} \\
&- 81X^{57}Y^{65} + 101X^{58}Y^{65} - 44X^{59}Y^{65} - 25X^{60}Y^{65} + 17X^{61}Y^{65} - 2X^{62}Y^{65} \\
&- X^{63}Y^{65} + X^{53}Y^{66} - 4X^{54}Y^{66} - 4X^{55}Y^{66} + 28X^{56}Y^{66} - 12X^{57}Y^{66} \\
&- 18X^{58}Y^{66} + 48X^{59}Y^{66} - 37X^{60}Y^{66} - 19X^{61}Y^{66} + 30X^{62}Y^{66} - 16X^{63}Y^{66} \\
&+ 3X^{64}Y^{66} + X^{54}Y^{67} - 2X^{55}Y^{67} - 7X^{56}Y^{67} + 26X^{57}Y^{67} - 53X^{58}Y^{67} \\
&+ 80X^{59}Y^{67} - 23X^{60}Y^{67} - 44X^{61}Y^{67} + 35X^{62}Y^{67} - 13X^{63}Y^{67} - X^{55}Y^{68} \\
&+ 2X^{56}Y^{68} + 4X^{57}Y^{68} - 14X^{58}Y^{68} - 8X^{59}Y^{68} + 54X^{60}Y^{68} - 24X^{61}Y^{68} \\
&- 32X^{62}Y^{68} + 45X^{63}Y^{68} - 28X^{64}Y^{68} + 4X^{65}Y^{68} - X^{56}Y^{69} - X^{57}Y^{69} \\
&+ 14X^{58}Y^{69} - 28X^{59}Y^{69} + 17X^{60}Y^{69} + 36X^{61}Y^{69} - 57X^{62}Y^{69} + 51X^{63}Y^{69} \\
&- 17X^{64}Y^{69} - 8X^{65}Y^{69} + 3X^{66}Y^{69} - 3X^{59}Y^{70} + 5X^{60}Y^{70} + 8X^{61}Y^{70} \\
&- 2X^{62}Y^{70} - 23X^{63}Y^{70} + 51X^{64}Y^{70} - 27X^{65}Y^{70} - 3X^{66}Y^{70} + 3X^{67}Y^{70} \\
&+ X^{58}Y^{71} + X^{59}Y^{71} - 16X^{60}Y^{71} + 12X^{61}Y^{71} + 29X^{62}Y^{71} - 59X^{63}Y^{71} \\
&+ 40X^{64}Y^{71} + 15X^{65}Y^{71} - 21X^{66}Y^{71} + 5X^{67}Y^{71} - X^{68}Y^{71} + 2X^{60}Y^{72}
\end{aligned}
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\begin{aligned}
&- 11X^{61}Y^{72} + 8X^{62}Y^{72} - 3X^{63}Y^{72} - 9X^{64}Y^{72} + 21X^{65}Y^{72} + 3X^{66}Y^{72} \\
&- 11X^{67}Y^{72} + 7X^{68}Y^{72} - 3X^{69}Y^{72} - X^{60}Y^{73} + X^{61}Y^{73} - 3X^{62}Y^{73} \\
&+ 14X^{63}Y^{73} - 37X^{64}Y^{73} + 33X^{65}Y^{73} + 14X^{66}Y^{73} - 28X^{67}Y^{73} + 16X^{68}Y^{73} \\
&+ X^{69}Y^{73} - 2X^{70}Y^{73} - 2X^{62}Y^{74} + 8X^{63}Y^{74} - 3X^{64}Y^{74} - 19X^{65}Y^{74} \\
&+ 21X^{66}Y^{74} + 7X^{67}Y^{74} - 16X^{68}Y^{74} + 9X^{69}Y^{74} - 2X^{63}Y^{75} + 4X^{64}Y^{75} \\
&- 7X^{65}Y^{75} - 7X^{66}Y^{75} + 25X^{67}Y^{75} - 34X^{68}Y^{75} + 25X^{69}Y^{75} - 5X^{70}Y^{75} \\
&+ 2X^{64}Y^{76} - 7X^{65}Y^{76} - 2X^{66}Y^{76} + 7X^{67}Y^{76} - X^{68}Y^{76} - 17X^{69}Y^{76} \\
&+ 19X^{70}Y^{76} - 2X^{71}Y^{76} - X^{72}Y^{76} + 3X^{65}Y^{77} - 5X^{66}Y^{77} - 8X^{67}Y^{77} \\
&+ 24X^{68}Y^{77} - 27X^{69}Y^{77} + 9X^{70}Y^{77} + 6X^{71}Y^{77} - X^{72}Y^{77} + 2X^{66}Y^{78} \\
&- 2X^{67}Y^{78} + 4X^{68}Y^{78} - 4X^{69}Y^{78} - 7X^{70}Y^{78} + 6X^{71}Y^{78} - 2X^{73}Y^{78} \\
&+ X^{74}Y^{78} - X^{68}Y^{79} + 11X^{69}Y^{79} - 22X^{70}Y^{79} + 2X^{71}Y^{79} + 10X^{72}Y^{79} \\
&- 6X^{73}Y^{79} + X^{74}Y^{79} - 2X^{68}Y^{80} + 2X^{69}Y^{80} + 4X^{70}Y^{80} - 11X^{71}Y^{80} \\
&+ 4X^{72}Y^{80} - X^{73}Y^{80} - X^{74}Y^{80} + X^{75}Y^{80} + 2X^{71}Y^{81} - 5X^{72}Y^{81} \\
&+ 9X^{73}Y^{81} - 11X^{74}Y^{81} + 4X^{75}Y^{81} + 2X^{70}Y^{82} - 7X^{72}Y^{82} + 10X^{73}Y^{82} \\
&- X^{74}Y^{82} - 6X^{75}Y^{82} + X^{76}Y^{82} + X^{72}Y^{83} - 4X^{73}Y^{83} + 5X^{74}Y^{83} \\
&- 3X^{75}Y^{83} - 2X^{76}Y^{83} + X^{72}Y^{84} - 3X^{73}Y^{84} + 8X^{74}Y^{84} - 7X^{75}Y^{84} \\
&- 2X^{76}Y^{84} + 3X^{77}Y^{84} - X^{78}Y^{84} - X^{74}Y^{85} + 5X^{75}Y^{85} - 4X^{77}Y^{85} \\
&+ 2X^{78}Y^{85} - X^{74}Y^{86} + 2X^{75}Y^{86} + 2X^{76}Y^{86} - 4X^{77}Y^{86} + 4X^{78}Y^{86} \\
&- 3X^{79}Y^{86} + X^{80}Y^{86} - X^{76}Y^{87} + X^{77}Y^{87} - 3X^{78}Y^{87} + 4X^{79}Y^{87} \\
&- 2X^{80}Y^{87} + 2X^{77}Y^{88} - 4X^{78}Y^{88} + 4X^{79}Y^{88} - X^{80}Y^{88} + X^{78}Y^{89} \\
&- X^{79}Y^{89} + 2X^{80}Y^{89} - 2X^{79}Y^{90} + 3X^{80}Y^{90} - X^{80}Y^{91} + X^{82}Y^{93} \\
&- X^{84}Y^{95}.
\end{aligned}
$$

$\zeta^{\triangleleft}_{M_3\times M_3}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{M_3\times M_3,p}(s)\big|_{p\to p^{-1}} = p^{28-18s}\zeta^{\triangleleft}_{M_3\times M_3,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{M_3\times M_3}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}
&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-2)\zeta_p(3s-4)^2\zeta_p(4s-4)\zeta_p(5s-5)\\
&\times\zeta_p(6s-5)\zeta_p(7s-5)\zeta_p(7s-6)\zeta_p(8s-6)\zeta_p(9s-7)\zeta_p(9s-10)\zeta_p(10s-10)\\
&\times\zeta_p(11s-11)\zeta_p(12s-12)\zeta_p(13s-15)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})\\
&\times W_4(p,p^{-s})W_5(p,p^{-s})W_6(p,p^{-s})W_7(p,p^{-s})
\end{aligned}$$

where

$$\begin{aligned}
W_1(X,Y) &= 1+X^{10}Y^9,\\
W_2(X,Y) &= 1+X^{14}Y^{13},\\
W_3(X,Y) &= 1-X^2Y^2+X^4Y^4+X^5Y^5-X^6Y^6-X^7Y^7+X^8Y^8+X^9Y^9\\
&\quad +X^{10}Y^{10}-X^{12}Y^{12}+X^{14}Y^{14},\\
W_4(X,Y) &= 1+2X^{25}Y^{28},\\
W_5(X,Y) &= 2+X^{11}Y^{14},\\
W_6(X,Y) &= 1+X^6Y^8,\\
W_7(X,Y) &= 1-X^4Y^9.
\end{aligned}$$

The ghost is unfriendly.

6 Natural boundary

$\zeta^{\triangleleft}_{M_3\times M_3}(s)$ has a natural boundary at $\Re(s)=10/9$, and is of type III.