The zeta function of M_3 counting all subrings

1 Presentation

M_3 has presentation

$$\langle z, x_1, x_2, x_3 \mid [z, x_1] = x_2, [z, x_2] = x_3 \rangle .$$

M_3 has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Gareth Taylor. It is

$$\zeta_{M_3,p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(2s-3)\zeta_p(3s-5)\zeta_p(4s-6)W(p, p^{-s})$$

where $W(X, Y)$ is

$$1 + X^2Y^2 + X^3Y^2 - X^3Y^3 + X^4Y^3 - X^5Y^4 + X^6Y^4 - X^6Y^5 - X^7Y^5 - X^9Y^7.$$

$\zeta_{M_3}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{M_3,p}(s)\big|_{p\to p^{-1}} = p^{6-4s}\zeta_{M_3,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{M_3}(s)$ is 2, with a quadruple pole at $s = 2$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(2s-3)\zeta_p(3s-5)\zeta_p(4s-6)W_1(p, p^{-s})W_2(p, p^{-s})$$

where

$$W_1(X, Y) = 1 + X^3Y^2 + X^6Y^4,$$
$$W_2(X, Y) = 1 - XY - X^3Y^3.$$

The ghost is unfriendly.

6 Natural boundary

$\zeta_{M_3}(s)$ has a natural boundary at $\Re(s) = 3/2$, and is of type III.