The zeta function of L_{W} counting ideals

1 Presentation

L_{W} has presentation

$$\langle z, w_1, w_2, x_1, x_2, y_1 \mid [z, w_1] = x_1, [z, w_2] = x_2, [z, x_1] = y_1 \rangle .$$

L_{W} has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{L_{\mathrm{W}},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(5s-6) \\ & \times \zeta_p(6s-6)\zeta_p(9s-11)W(p, p^{-s})\end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned}&1 + X^3Y^3 - X^4Y^5 - X^6Y^7 - X^7Y^7 + X^8Y^7 - X^8Y^8 - X^9Y^9 - X^{10}Y^9 \\ &+ X^{10}Y^{10} - X^{11}Y^{10} + X^{10}Y^{11} - X^{11}Y^{11} + X^{11}Y^{12} - X^{14}Y^{12} + X^{13}Y^{13} \\ &- X^{14}Y^{13} + X^{14}Y^{14} + X^{15}Y^{14} + X^{17}Y^{16} + X^{18}Y^{17} + X^{20}Y^{18} - X^{21}Y^{21} \\ &- X^{24}Y^{23}.\end{aligned}$$

$\zeta^{\triangleleft}_{L_{\mathrm{W}}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies no functional equation.

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{L_{\mathrm{W}}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(5s-6)\zeta_p(6s-6) \\ &\times \zeta_p(9s-11)W_1(p, p^{-s})W_2(p, p^{-s})W_3(p, p^{-s})\end{aligned}$$

where

$$W_1(X,Y) = 1 - X^{14}Y^{12},$$
$$W_2(X,Y) = -1 + X^6Y^6,$$
$$W_3(X,Y) = 1 - X^4Y^5.$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{L_{\mathrm{W}}}(s)$ has a natural boundary at $\Re(s) = 7/6$, and is of type III.

7 Notes

This was the first local ideal zeta function calculated which satisfied no functional equation. Another six local ideal zeta functions without functional equations have since been calculated.

No local zeta functions counting all subrings have yet been discovered. Conjecturally there aren't any.