The zeta function of $L_{3,3}$ counting ideals

1 Presentation

$L_{3,3}$ has presentation

$$\langle z, w_1, w_2, x_1, x_2, y_1, y_2 \mid [z, w_1] = x_1, [z, w_2] = x_2, [z, x_1] = y_1, [z, x_2] = y_2 \rangle .$$

$L_{3,3}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{L_{3,3},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-4)\zeta_p(4s-5)\zeta_p(5s-6)\zeta_p(6s-7) \\ & \times \zeta_p(7s-6)\zeta_p(8s-10)\zeta_p(9s-12)\zeta_p(11s-12)\zeta_p(4s-4)^{-1} \\ & \times W(p, p^{-s})\end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned}
&1 + X^3Y^3 + 2X^4Y^4 - X^4Y^5 + X^6Y^5 + X^6Y^6 - X^6Y^7 + X^9Y^7 - X^6Y^8 \\
&+ 2X^8Y^8 - X^8Y^9 - X^{10}Y^9 - X^9Y^{10} + X^{12}Y^{10} - X^{10}Y^{11} - X^{12}Y^{11} \\
&- X^{13}Y^{12} - X^{12}Y^{13} - X^{14}Y^{13} - 2X^{16}Y^{13} - 2X^{15}Y^{14} - X^{14}Y^{15} - X^{16}Y^{15} \\
&- X^{18}Y^{15} + 2X^{16}Y^{16} - X^{18}Y^{16} - X^{19}Y^{16} - X^{18}Y^{17} - 2X^{20}Y^{17} + X^{18}Y^{18} \\
&+ X^{20}Y^{18} - X^{21}Y^{18} + X^{19}Y^{19} - X^{20}Y^{19} - X^{22}Y^{19} + 2X^{20}Y^{20} + X^{22}Y^{20} \\
&+ X^{21}Y^{21} + X^{22}Y^{21} - 2X^{24}Y^{21} + X^{22}Y^{22} + X^{24}Y^{22} + X^{26}Y^{22} + 2X^{25}Y^{23} \\
&+ 2X^{24}Y^{24} + X^{26}Y^{24} + X^{28}Y^{24} + X^{27}Y^{25} + X^{28}Y^{26} + X^{30}Y^{26} - X^{28}Y^{27} \\
&+ X^{31}Y^{27} + X^{30}Y^{28} + X^{32}Y^{28} - 2X^{32}Y^{29} + X^{34}Y^{29} - X^{31}Y^{30} + X^{34}Y^{30} \\
&- X^{34}Y^{31} - X^{34}Y^{32} + X^{36}Y^{32} - 2X^{36}Y^{33} - X^{37}Y^{34} - X^{40}Y^{37}.
\end{aligned}$$

$\zeta^{\triangleleft}_{L_{3,3}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{L_{3,3},p}(s)\Big|_{p \to p^{-1}} = -p^{21-15s}\zeta^{\triangleleft}_{L_{3,3},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{L_{3,3}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}&\zeta_p(s)\zeta_p(s-2)\zeta_p(3s-4)\zeta_p(4s-5)\zeta_p(5s-6)\zeta_p(6s-7)\zeta_p(7s-6)\zeta_p(8s-10)\\&\times \zeta_p(9s-12)\zeta_p(11s-12)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})W_4(p,p^{-s})\\&\times W_5(p,p^{-s})\end{aligned}$$

where

$$\begin{aligned}W_1(X,Y) &= 1 + X^9Y^7,\\W_2(X,Y) &= 1 - 2X^7Y^6,\\W_3(X,Y) &= -2 + X^{18}Y^{16},\\W_4(X,Y) &= 1 + XY + X^2Y^2 + X^3Y^3,\\W_5(X,Y) &= 1 - X^6Y^8.\end{aligned}$$

The ghost is unfriendly.

6 Natural boundary

$\zeta^{\triangleleft}_{L_{3,3}}(s)$ has a natural boundary at $\Re(s) = 9/7$, and is of type III.