

The zeta function of $H \times \mathfrak{g}_{6,12}$ counting ideals

1 Presentation

$H \times \mathfrak{g}_{6,12}$ has presentation

$$\langle t, u, x_1, x_2, x_3, x_4, v, x_5, x_6 \mid [t, u] = v, [x_1, x_3] = x_5, [x_1, x_5] = x_6, [x_2, x_4] = x_6 \rangle.$$

$H \times \mathfrak{g}_{6,12}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta_{H \times \mathfrak{g}_{6,12}, p}^{\triangleleft}(s) &= \zeta_p(s) \zeta_p(s-1) \zeta_p(s-2) \zeta_p(s-3) \zeta_p(s-4) \zeta_p(s-5) \zeta_p(3s-6)^2 \\ &\quad \times \zeta_p(5s-7) \zeta_p(6s-6) \zeta_p(7s-7) \zeta_p(8s-7) \zeta_p(9s-8) \\ &\quad \times \zeta_p(11s-14) W(p, p^{-s}) \end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned} &1 - X^6 Y^5 - X^6 Y^7 - X^6 Y^8 + X^6 Y^9 - 2X^7 Y^9 + X^{12} Y^{11} - 2X^{13} Y^{11} \\ &+ 2X^{13} Y^{12} - X^{14} Y^{12} + 2X^{13} Y^{13} - X^{14} Y^{13} + X^{14} Y^{14} + 2X^{13} Y^{15} \\ &- X^{14} Y^{15} + X^{14} Y^{16} + X^{20} Y^{16} + X^{14} Y^{17} + X^{20} Y^{18} + X^{20} Y^{19} - 2X^{19} Y^{20} \\ &+ 2X^{21} Y^{20} - X^{20} Y^{21} - X^{20} Y^{22} - X^{26} Y^{23} - X^{20} Y^{24} - X^{26} Y^{24} + X^{26} Y^{25} \\ &- 2X^{27} Y^{25} - X^{26} Y^{26} + X^{26} Y^{27} - 2X^{27} Y^{27} + X^{26} Y^{28} - 2X^{27} Y^{28} \\ &+ 2X^{27} Y^{29} - X^{28} Y^{29} + 2X^{33} Y^{31} - X^{34} Y^{31} + X^{34} Y^{32} + X^{34} Y^{33} + X^{34} Y^{35} \\ &- X^{40} Y^{40}. \end{aligned}$$

$\zeta_{H \times \mathfrak{g}_{6,12}}^{\triangleleft}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{H \times \mathfrak{g}_{6,12}, p}^{\triangleleft}(s) \Big|_{p \rightarrow p^{-1}} = -p^{36-18s} \zeta_{H \times \mathfrak{g}_{6,12}, p}^{\triangleleft}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{H \times \mathfrak{g}_{6,12}}^{\triangleleft}(s)$ is 6, with a simple pole at $s = 6$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(s-5)\zeta_p(3s-6)^2\zeta_p(5s-7) \\ & \times \zeta_p(6s-6)\zeta_p(7s-7)\zeta_p(8s-7)\zeta_p(9s-8)\zeta_p(11s-14)W_1(p,p^{-s})W_2(p,p^{-s}) \\ & \times W_3(p,p^{-s}) \end{aligned}$$

where

$$\begin{aligned} W_1(X, Y) &= 1 + X^{20}Y^{16}, \\ W_2(X, Y) &= 1 - X^{14}Y^{15}, \\ W_3(X, Y) &= -1 - X^6Y^9. \end{aligned}$$

The ghost is friendly.

6 Natural boundary

$\zeta_{H \times \mathfrak{g}_{6,12}}^{\triangleleft}(s)$ has a natural boundary at $\Re(s) = 5/4$, and is of type III.