The zeta function of $H \times \mathbb{Z}^3$ counting all subrings

1 Presentation

$H \times \mathbb{Z}^3$ has presentation

$$\langle x, y, a, b, c, z \mid [x, y] = z \rangle .$$

$H \times \mathbb{Z}^3$ has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta_{H\times\mathbb{Z}^3,p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(2s-5)\zeta_p(2s-6) \times \zeta_p(3s-6)^{-1}.$$

$\zeta_{H\times\mathbb{Z}^3}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{H\times\mathbb{Z}^3,p}(s)\big|_{p\to p^{-1}} = p^{15-6s}\zeta_{H\times\mathbb{Z}^3,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{H\times\mathbb{Z}^3}(s)$ is 5, with a simple pole at $s = 5$.

5 Ghost zeta function

This zeta function is its own ghost.

6 Natural boundary

$\zeta_{H\times\mathbb{Z}^3}(s)$ has meromorphic continuation to the whole of $\mathbb{C}$.