

The zeta function of $H \times \mathbb{Z}^2$ counting all subrings

1 Presentation

$H \times \mathbb{Z}^2$ has presentation

$$\langle x, y, z, a, b \mid [x, y] = z \rangle.$$

$H \times \mathbb{Z}^2$ has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta_{H \times \mathbb{Z}^2, p}(s) = \zeta_p(s) \zeta_p(s-1) \zeta_p(s-2) \zeta_p(s-3) \zeta_p(2s-4) \zeta_p(2s-5) \\ \times \zeta_p(3s-5)^{-1}.$$

$\zeta_{H \times \mathbb{Z}^2}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{H \times \mathbb{Z}^2, p}(s) \Big|_{p \rightarrow p^{-1}} = -p^{10-5s} \zeta_{H \times \mathbb{Z}^2, p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{H \times \mathbb{Z}^2}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

This zeta function is its own ghost.

6 Natural boundary

$\zeta_{H \times \mathbb{Z}^2}(s)$ has meromorphic continuation to the whole of \mathbb{C} .