The zeta function of $H \times M_3$ counting ideals

1 Presentation

$H \times M_3$ has presentation

$$\langle t, z, u, x_1, v, x_2, x_3 \mid [t, u] = v, [z, x_1] = x_2, [z, x_2] = x_3 \rangle .$$

$H \times M_3$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{H\times M_3,p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)^2\zeta_p(4s-4)\zeta_p(5s-5)\\ & \times \zeta_p(6s-5)\zeta_p(7s-6)\zeta_p(9s-10)W(p,p^{-s})\end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned}&1 - 2X^4Y^5 + X^5Y^5 - X^4Y^6 + X^4Y^7 - 2X^5Y^7 + X^8Y^9 - 2X^9Y^9 + 3X^9Y^{11}\\ &- 2X^{10}Y^{11} + X^9Y^{12} + X^{10}Y^{13} + X^{13}Y^{14} + X^{14}Y^{15} - 2X^{13}Y^{16} + 3X^{14}Y^{16}\\ &- 2X^{14}Y^{18} + X^{15}Y^{18} - 2X^{18}Y^{20} + X^{19}Y^{20} - X^{19}Y^{21} + X^{18}Y^{22}\\ &- 2X^{19}Y^{22} + X^{23}Y^{27}.\end{aligned}$$

$\zeta^{\triangleleft}_{H\times M_3}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{H\times M_3,p}(s)\Big|_{p\to p^{-1}} = -p^{21-14s}\zeta^{\triangleleft}_{H\times M_3,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{H\times M_3}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)^2\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(6s-5)$$
$$\times\, \zeta_p(7s-6)\zeta_p(9s-10)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})$$

where

$$\begin{aligned}W_1(X,Y) &= 1+X^5Y^5-2X^9Y^9,\\ W_2(X,Y) &= -2+X^{10}Y^{11},\\ W_3(X,Y) &= 1+X^4Y^7.\end{aligned}$$

The ghost is unfriendly.

6 Natural boundary

$\zeta^{\triangleleft}_{H\times M_3}(s)$ has a natural boundary at $\Re(s)=1$, and is of type I.