

The zeta function of $H \times H \times M_3$ counting ideals

1 Presentation

$H \times H \times M_3$ has presentation

$$\left\langle z, x_1, t_1, t_2, u_1, u_2, x_2, v_1, v_2, x_3 \left| \begin{array}{l} [z, x_1] = x_2, [z, x_2] = x_3, \\ [t_1, u_1] = v_1, [t_2, u_2] = v_2 \end{array} \right. \right\rangle.$$

$H \times H \times M_3$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta_{H \times H \times M_3, p}^{\triangleleft}(s) &= \zeta_p(s) \zeta_p(s-1) \zeta_p(s-2) \zeta_p(s-3) \zeta_p(s-4) \zeta_p(s-5) \zeta_p(3s-6)^3 \\ &\quad \times \zeta_p(4s-6) \zeta_p(5s-7) \zeta_p(6s-7) \zeta_p(7s-8) \zeta_p(8s-8) \\ &\quad \times \zeta_p(8s-14) \zeta_p(9s-9) \zeta_p(9s-14) \zeta_p(10s-15) \zeta_p(11s-16) \\ &\quad \times \zeta_p(12s-21) W(p, p^{-s}) \end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned} &1 - 4X^6Y^5 + 3X^7Y^5 - 2X^6Y^6 + X^7Y^6 + 3X^6Y^7 - 6X^7Y^7 + 2X^8Y^7 + X^6Y^8 \\ &- 2X^7Y^8 + X^{12}Y^8 - 2X^{13}Y^8 - X^6Y^9 + 3X^7Y^9 - 3X^8Y^9 + 3X^{12}Y^9 \\ &- 6X^{13}Y^9 + 2X^{14}Y^9 + X^{13}Y^{10} - 2X^{14}Y^{10} - X^{12}Y^{11} + 10X^{13}Y^{11} \\ &- 9X^{14}Y^{11} + 4X^{13}Y^{12} - 4X^{14}Y^{12} + X^{15}Y^{12} - X^{18}Y^{12} + 3X^{19}Y^{12} \\ &- 3X^{20}Y^{12} - X^{13}Y^{13} + 8X^{14}Y^{13} - 5X^{15}Y^{13} + 2X^{19}Y^{13} - X^{21}Y^{13} \\ &- 2X^{13}Y^{14} + 9X^{14}Y^{14} - 5X^{15}Y^{14} + X^{19}Y^{14} + 4X^{20}Y^{14} - 3X^{21}Y^{14} \\ &- X^{13}Y^{15} - X^{14}Y^{15} + 7X^{15}Y^{15} - 3X^{16}Y^{15} - 3X^{19}Y^{15} + 8X^{20}Y^{15} \\ &- 2X^{22}Y^{15} - 2X^{14}Y^{16} + 3X^{15}Y^{16} - 11X^{19}Y^{16} + 19X^{20}Y^{16} - 4X^{22}Y^{16} \\ &- X^{15}Y^{17} + 2X^{16}Y^{17} - X^{19}Y^{17} - 5X^{20}Y^{17} + 11X^{21}Y^{17} - X^{23}Y^{17} \\ &- 3X^{25}Y^{17} + 5X^{26}Y^{17} + 3X^{27}Y^{17} - 3X^{28}Y^{17} + 6X^{19}Y^{18} - 26X^{20}Y^{18} \\ &+ 21X^{21}Y^{18} + 4X^{22}Y^{18} - 3X^{23}Y^{18} - X^{25}Y^{18} + 2X^{26}Y^{18} + X^{28}Y^{18} \\ &- X^{20}Y^{19} - 7X^{21}Y^{19} + 8X^{22}Y^{19} + X^{23}Y^{19} + 6X^{25}Y^{19} - 22X^{26}Y^{19} \\ &+ 13X^{27}Y^{19} + 8X^{28}Y^{19} - 3X^{29}Y^{19} + 7X^{20}Y^{20} - 24X^{21}Y^{20} + 15X^{22}Y^{20} \\ &+ X^{23}Y^{20} + 2X^{25}Y^{20} - 14X^{26}Y^{20} + 8X^{27}Y^{20} + 6X^{28}Y^{20} - 2X^{29}Y^{20} \\ &- X^{33}Y^{20} + 2X^{34}Y^{20} - X^{21}Y^{21} - 3X^{22}Y^{21} + 2X^{23}Y^{21} - 3X^{25}Y^{21} \end{aligned}$$

$$\begin{aligned}
& + 18X^{26}Y^{21} - 38X^{27}Y^{21} + 14X^{28}Y^{21} + 8X^{29}Y^{21} - X^{30}Y^{21} - 2X^{32}Y^{21} \\
& + 3X^{33}Y^{21} + 5X^{21}Y^{22} - 14X^{22}Y^{22} + 7X^{23}Y^{22} + 11X^{26}Y^{22} - 28X^{27}Y^{22} \\
& + 2X^{28}Y^{22} + 11X^{29}Y^{22} - 2X^{30}Y^{22} + X^{32}Y^{22} - 7X^{33}Y^{22} + 6X^{34}Y^{22} \\
& - X^{22}Y^{23} - 3X^{26}Y^{23} + 20X^{27}Y^{23} - 40X^{28}Y^{23} + 14X^{29}Y^{23} + 2X^{30}Y^{23} \\
& + 7X^{32}Y^{23} - 20X^{33}Y^{23} + 6X^{34}Y^{23} + 5X^{35}Y^{23} + 3X^{22}Y^{24} - 5X^{23}Y^{24} \\
& + X^{24}Y^{24} - 2X^{26}Y^{24} + 23X^{27}Y^{24} - 37X^{28}Y^{24} + 3X^{29}Y^{24} + 6X^{30}Y^{24} \\
& + X^{32}Y^{24} + 12X^{33}Y^{24} - 36X^{34}Y^{24} + 19X^{35}Y^{24} - X^{36}Y^{24} + 17X^{28}Y^{25} \\
& - 33X^{29}Y^{25} + 8X^{30}Y^{25} + 2X^{31}Y^{25} - 5X^{32}Y^{25} + 27X^{33}Y^{25} - 36X^{34}Y^{25} \\
& + 7X^{36}Y^{25} + X^{39}Y^{25} - 5X^{40}Y^{25} + 3X^{41}Y^{25} - 4X^{27}Y^{26} + 27X^{28}Y^{26} \\
& - 29X^{29}Y^{26} + X^{30}Y^{26} + 2X^{31}Y^{26} - X^{32}Y^{26} + 2X^{33}Y^{26} + 29X^{34}Y^{26} \\
& - 56X^{35}Y^{26} + 18X^{36}Y^{26} + X^{37}Y^{26} + 3X^{39}Y^{26} - 10X^{40}Y^{26} + 5X^{41}Y^{26} \\
& - X^{28}Y^{27} + 12X^{29}Y^{27} - 12X^{30}Y^{27} + X^{32}Y^{27} - 8X^{33}Y^{27} + 47X^{34}Y^{27} \\
& - 52X^{35}Y^{27} + 4X^{36}Y^{27} + 3X^{37}Y^{27} - 2X^{39}Y^{27} + 13X^{40}Y^{27} - 21X^{41}Y^{27} \\
& + 4X^{42}Y^{27} + 2X^{43}Y^{27} - 4X^{28}Y^{28} + 14X^{29}Y^{28} - 8X^{30}Y^{28} - 2X^{31}Y^{28} \\
& - 3X^{33}Y^{28} + 9X^{34}Y^{28} + 31X^{35}Y^{28} - 46X^{36}Y^{28} + 7X^{37}Y^{28} + X^{38}Y^{28} \\
& + 24X^{40}Y^{28} - 39X^{41}Y^{28} + 9X^{42}Y^{28} + 2X^{43}Y^{28} + X^{29}Y^{29} + X^{30}Y^{29} \\
& - X^{31}Y^{29} - 2X^{33}Y^{29} - 11X^{34}Y^{29} + 62X^{35}Y^{29} - 51X^{36}Y^{29} + 4X^{37}Y^{29} \\
& + X^{38}Y^{29} + X^{39}Y^{29} - 4X^{40}Y^{29} + 23X^{41}Y^{29} - 24X^{42}Y^{29} + X^{44}Y^{29} \\
& + 3X^{46}Y^{29} - 5X^{47}Y^{29} + X^{48}Y^{29} - X^{29}Y^{30} + 3X^{30}Y^{30} - X^{31}Y^{30} \\
& - 11X^{34}Y^{30} + 19X^{35}Y^{30} + 22X^{36}Y^{30} - 26X^{37}Y^{30} + 2X^{38}Y^{30} - 6X^{39}Y^{30} \\
& - 4X^{40}Y^{30} + 69X^{41}Y^{30} - 69X^{42}Y^{30} + 11X^{43}Y^{30} + X^{44}Y^{30} + X^{47}Y^{30} \\
& - X^{48}Y^{30} - X^{49}Y^{30} - 3X^{34}Y^{31} - 9X^{35}Y^{31} + 43X^{36}Y^{31} - 26X^{37}Y^{31} \\
& + 2X^{38}Y^{31} - X^{39}Y^{31} - 5X^{40}Y^{31} + 18X^{41}Y^{31} + 16X^{42}Y^{31} - 26X^{43}Y^{31} \\
& + 4X^{44}Y^{31} - 5X^{46}Y^{31} + 20X^{47}Y^{31} - 16X^{48}Y^{31} + 2X^{34}Y^{32} - 11X^{35}Y^{32} \\
& + 10X^{36}Y^{32} + 13X^{37}Y^{32} - 8X^{38}Y^{32} + 3X^{39}Y^{32} - 14X^{40}Y^{32} - 25X^{41}Y^{32} \\
& + 112X^{42}Y^{32} - 71X^{43}Y^{32} + 6X^{44}Y^{32} + X^{46}Y^{32} + 7X^{47}Y^{32} - X^{48}Y^{32} \\
& - 7X^{49}Y^{32} + 2X^{50}Y^{32} - 5X^{35}Y^{33} - 2X^{36}Y^{33} + 21X^{37}Y^{33} - 10X^{38}Y^{33} \\
& + 4X^{40}Y^{33} - 28X^{41}Y^{33} + 32X^{42}Y^{33} + 24X^{43}Y^{33} - 22X^{44}Y^{33} + 2X^{45}Y^{33} \\
& - 2X^{46}Y^{33} - 10X^{47}Y^{33} + 46X^{48}Y^{33} - 31X^{49}Y^{33} + 2X^{50}Y^{33} + X^{35}Y^{34} \\
& - 4X^{36}Y^{34} + 2X^{37}Y^{34} + 4X^{38}Y^{34} - X^{39}Y^{34} + 5X^{40}Y^{34} - 16X^{41}Y^{34} \\
& - 33X^{42}Y^{34} + 93X^{43}Y^{34} - 39X^{44}Y^{34} + X^{45}Y^{34} - 3X^{46}Y^{34} - 11X^{47}Y^{34} \\
& + 32X^{48}Y^{34} + 5X^{49}Y^{34} - 17X^{50}Y^{34} + 3X^{51}Y^{34} + X^{53}Y^{34} + X^{54}Y^{34} \\
& - X^{55}Y^{34} - 3X^{36}Y^{35} - 2X^{37}Y^{35} + 5X^{38}Y^{35} - X^{40}Y^{35} - 22X^{42}Y^{35} \\
& + 23X^{43}Y^{35} + 13X^{44}Y^{35} - 7X^{45}Y^{35} + 3X^{46}Y^{35} - 14X^{47}Y^{35} - 21X^{48}Y^{35}
\end{aligned}$$

$$\begin{aligned}
& + 77X^{49}Y^{35} - 36X^{50}Y^{35} + X^{51}Y^{35} - X^{53}Y^{35} + 3X^{54}Y^{35} + 2X^{55}Y^{35} \\
& - 2X^{56}Y^{35} + 7X^{41}Y^{36} - 22X^{42}Y^{36} - 20X^{43}Y^{36} + 51X^{44}Y^{36} - 12X^{45}Y^{36} \\
& - X^{47}Y^{36} - 39X^{48}Y^{36} + 53X^{49}Y^{36} + 4X^{50}Y^{36} - 9X^{51}Y^{36} - 7X^{53}Y^{36} \\
& + 6X^{54}Y^{36} + 14X^{55}Y^{36} - 9X^{56}Y^{36} + 2X^{41}Y^{37} - 4X^{42}Y^{37} - 20X^{43}Y^{37} \\
& + 20X^{44}Y^{37} + 10X^{47}Y^{37} - 32X^{48}Y^{37} - 28X^{49}Y^{37} + 80X^{50}Y^{37} - 26X^{51}Y^{37} \\
& - X^{53}Y^{37} - 8X^{54}Y^{37} + 16X^{55}Y^{37} + X^{56}Y^{37} - 3X^{57}Y^{37} + 7X^{42}Y^{38} \\
& - 13X^{43}Y^{38} - 11X^{44}Y^{38} + 14X^{45}Y^{38} + 7X^{47}Y^{38} - 6X^{48}Y^{38} - 58X^{49}Y^{38} \\
& + 60X^{50}Y^{38} - 4X^{51}Y^{38} - X^{52}Y^{38} + 7X^{53}Y^{38} - 25X^{54}Y^{38} - 7X^{55}Y^{38} \\
& + 44X^{56}Y^{38} - 13X^{57}Y^{38} - X^{58}Y^{38} + 3X^{42}Y^{39} - X^{43}Y^{39} - 14X^{44}Y^{39} \\
& + 9X^{45}Y^{39} + 22X^{48}Y^{39} - 46X^{49}Y^{39} - 30X^{50}Y^{39} + 59X^{51}Y^{39} - 13X^{52}Y^{39} \\
& + 2X^{53}Y^{39} - 6X^{54}Y^{39} - 30X^{55}Y^{39} + 43X^{56}Y^{39} - 8X^{57}Y^{39} - X^{60}Y^{39} \\
& - X^{61}Y^{39} + 4X^{62}Y^{39} - X^{63}Y^{39} + 2X^{43}Y^{40} - 3X^{44}Y^{40} - 3X^{45}Y^{40} \\
& + 2X^{46}Y^{40} - 2X^{47}Y^{40} + 15X^{48}Y^{40} - 2X^{49}Y^{40} - 60X^{50}Y^{40} + 41X^{51}Y^{40} \\
& - 2X^{52}Y^{40} - 2X^{53}Y^{40} + 19X^{54}Y^{40} - 36X^{55}Y^{40} - 26X^{56}Y^{40} + 51X^{57}Y^{40} \\
& - 10X^{58}Y^{40} - 3X^{60}Y^{40} - 2X^{61}Y^{40} + 5X^{62}Y^{40} + X^{44}Y^{41} - 2X^{45}Y^{41} \\
& - X^{48}Y^{41} + 26X^{49}Y^{41} - 35X^{50}Y^{41} - 25X^{51}Y^{41} + 27X^{52}Y^{41} - 3X^{53}Y^{41} \\
& + 16X^{54}Y^{41} - 20X^{55}Y^{41} - 54X^{56}Y^{41} + 57X^{57}Y^{41} - 8X^{58}Y^{41} - X^{59}Y^{41} \\
& + 2X^{60}Y^{41} - 4X^{61}Y^{41} - 9X^{62}Y^{41} + 10X^{63}Y^{41} - 3X^{48}Y^{42} + 12X^{49}Y^{42} \\
& - X^{50}Y^{42} - 25X^{51}Y^{42} + 9X^{52}Y^{42} - 2X^{54}Y^{42} + 37X^{55}Y^{42} - 58X^{56}Y^{42} \\
& - 25X^{57}Y^{42} + 41X^{58}Y^{42} - 6X^{59}Y^{42} + 7X^{60}Y^{42} - 13X^{61}Y^{42} - 14X^{62}Y^{42} \\
& + 17X^{63}Y^{42} - X^{64}Y^{42} - X^{49}Y^{43} + 17X^{50}Y^{43} - 14X^{51}Y^{43} - 13X^{52}Y^{43} \\
& + 7X^{53}Y^{43} - 6X^{54}Y^{43} + 41X^{55}Y^{43} - 25X^{56}Y^{43} - 58X^{57}Y^{43} + 37X^{58}Y^{43} \\
& - 2X^{59}Y^{43} + 9X^{61}Y^{43} - 25X^{62}Y^{43} - X^{63}Y^{43} + 12X^{64}Y^{43} - 3X^{65}Y^{43} \\
& + 10X^{50}Y^{44} - 9X^{51}Y^{44} - 4X^{52}Y^{44} + 2X^{53}Y^{44} - X^{54}Y^{44} - 8X^{55}Y^{44} \\
& + 57X^{56}Y^{44} - 54X^{57}Y^{44} - 20X^{58}Y^{44} + 16X^{59}Y^{44} - 3X^{60}Y^{44} + 27X^{61}Y^{44} \\
& - 25X^{62}Y^{44} - 35X^{63}Y^{44} + 26X^{64}Y^{44} - X^{65}Y^{44} - 2X^{68}Y^{44} + X^{69}Y^{44} \\
& + 5X^{51}Y^{45} - 2X^{52}Y^{45} - 3X^{53}Y^{45} - 10X^{55}Y^{45} + 51X^{56}Y^{45} - 26X^{57}Y^{45} \\
& - 36X^{58}Y^{45} + 19X^{59}Y^{45} - 2X^{60}Y^{45} - 2X^{61}Y^{45} + 41X^{62}Y^{45} - 60X^{63}Y^{45} \\
& - 2X^{64}Y^{45} + 15X^{65}Y^{45} - 2X^{66}Y^{45} + 2X^{67}Y^{45} - 3X^{68}Y^{45} - 3X^{69}Y^{45} \\
& + 2X^{70}Y^{45} - X^{50}Y^{46} + 4X^{51}Y^{46} - X^{52}Y^{46} - X^{53}Y^{46} - 8X^{56}Y^{46} \\
& + 43X^{57}Y^{46} - 30X^{58}Y^{46} - 6X^{59}Y^{46} + 2X^{60}Y^{46} - 13X^{61}Y^{46} + 59X^{62}Y^{46} \\
& - 30X^{63}Y^{46} - 46X^{64}Y^{46} + 22X^{65}Y^{46} + 9X^{68}Y^{46} - 14X^{69}Y^{46} - X^{70}Y^{46} \\
& + 3X^{71}Y^{46} - X^{55}Y^{47} - 13X^{56}Y^{47} + 44X^{57}Y^{47} - 7X^{58}Y^{47} - 25X^{59}Y^{47} \\
& + 7X^{60}Y^{47} - X^{61}Y^{47} - 4X^{62}Y^{47} + 60X^{63}Y^{47} - 58X^{64}Y^{47} - 6X^{65}Y^{47}
\end{aligned}$$

$$\begin{aligned}
& + 7X^{66}Y^{47} + 14X^{68}Y^{47} - 11X^{69}Y^{47} - 13X^{70}Y^{47} + 7X^{71}Y^{47} - 3X^{56}Y^{48} \\
& + X^{57}Y^{48} + 16X^{58}Y^{48} - 8X^{59}Y^{48} - X^{60}Y^{48} - 26X^{62}Y^{48} + 80X^{63}Y^{48} \\
& - 28X^{64}Y^{48} - 32X^{65}Y^{48} + 10X^{66}Y^{48} + 20X^{69}Y^{48} - 20X^{70}Y^{48} - 4X^{71}Y^{48} \\
& + 2X^{72}Y^{48} - 9X^{57}Y^{49} + 14X^{58}Y^{49} + 6X^{59}Y^{49} - 7X^{60}Y^{49} - 9X^{62}Y^{49} \\
& + 4X^{63}Y^{49} + 53X^{64}Y^{49} - 39X^{65}Y^{49} - X^{66}Y^{49} - 12X^{68}Y^{49} + 51X^{69}Y^{49} \\
& - 20X^{70}Y^{49} - 22X^{71}Y^{49} + 7X^{72}Y^{49} - 2X^{57}Y^{50} + 2X^{58}Y^{50} + 3X^{59}Y^{50} \\
& - X^{60}Y^{50} + X^{62}Y^{50} - 36X^{63}Y^{50} + 77X^{64}Y^{50} - 21X^{65}Y^{50} - 14X^{66}Y^{50} \\
& + 3X^{67}Y^{50} - 7X^{68}Y^{50} + 13X^{69}Y^{50} + 23X^{70}Y^{50} - 22X^{71}Y^{50} - X^{73}Y^{50} \\
& + 5X^{75}Y^{50} - 2X^{76}Y^{50} - 3X^{77}Y^{50} - X^{58}Y^{51} + X^{59}Y^{51} + X^{60}Y^{51} \\
& + 3X^{62}Y^{51} - 17X^{63}Y^{51} + 5X^{64}Y^{51} + 32X^{65}Y^{51} - 11X^{66}Y^{51} - 3X^{67}Y^{51} \\
& + X^{68}Y^{51} - 39X^{69}Y^{51} + 93X^{70}Y^{51} - 33X^{71}Y^{51} - 16X^{72}Y^{51} + 5X^{73}Y^{51} \\
& - X^{74}Y^{51} + 4X^{75}Y^{51} + 2X^{76}Y^{51} - 4X^{77}Y^{51} + X^{78}Y^{51} + 2X^{63}Y^{52} \\
& - 31X^{64}Y^{52} + 46X^{65}Y^{52} - 10X^{66}Y^{52} - 2X^{67}Y^{52} + 2X^{68}Y^{52} - 22X^{69}Y^{52} \\
& + 24X^{70}Y^{52} + 32X^{71}Y^{52} - 28X^{72}Y^{52} + 4X^{73}Y^{52} - 10X^{75}Y^{52} + 21X^{76}Y^{52} \\
& - 2X^{77}Y^{52} - 5X^{78}Y^{52} + 2X^{63}Y^{53} - 7X^{64}Y^{53} - X^{65}Y^{53} + 7X^{66}Y^{53} \\
& + X^{67}Y^{53} + 6X^{69}Y^{53} - 71X^{70}Y^{53} + 112X^{71}Y^{53} - 25X^{72}Y^{53} - 14X^{73}Y^{53} \\
& + 3X^{74}Y^{53} - 8X^{75}Y^{53} + 13X^{76}Y^{53} + 10X^{77}Y^{53} - 11X^{78}Y^{53} + 2X^{79}Y^{53} \\
& - 16X^{65}Y^{54} + 20X^{66}Y^{54} - 5X^{67}Y^{54} + 4X^{69}Y^{54} - 26X^{70}Y^{54} + 16X^{71}Y^{54} \\
& + 18X^{72}Y^{54} - 5X^{73}Y^{54} - X^{74}Y^{54} + 2X^{75}Y^{54} - 26X^{76}Y^{54} + 43X^{77}Y^{54} \\
& - 9X^{78}Y^{54} - 3X^{79}Y^{54} - X^{64}Y^{55} - X^{65}Y^{55} + X^{66}Y^{55} + X^{69}Y^{55} \\
& + 11X^{70}Y^{55} - 69X^{71}Y^{55} + 69X^{72}Y^{55} - 4X^{73}Y^{55} - 6X^{74}Y^{55} + 2X^{75}Y^{55} \\
& - 26X^{76}Y^{55} + 22X^{77}Y^{55} + 19X^{78}Y^{55} - 11X^{79}Y^{55} - X^{82}Y^{55} + 3X^{83}Y^{55} \\
& - X^{84}Y^{55} + X^{65}Y^{56} - 5X^{66}Y^{56} + 3X^{67}Y^{56} + X^{69}Y^{56} - 24X^{71}Y^{56} \\
& + 23X^{72}Y^{56} - 4X^{73}Y^{56} + X^{74}Y^{56} + X^{75}Y^{56} + 4X^{76}Y^{56} - 51X^{77}Y^{56} \\
& + 62X^{78}Y^{56} - 11X^{79}Y^{56} - 2X^{80}Y^{56} - X^{82}Y^{56} + X^{83}Y^{56} + X^{84}Y^{56} \\
& + 2X^{70}Y^{57} + 9X^{71}Y^{57} - 39X^{72}Y^{57} + 24X^{73}Y^{57} + X^{75}Y^{57} + 7X^{76}Y^{57} \\
& - 46X^{77}Y^{57} + 31X^{78}Y^{57} + 9X^{79}Y^{57} - 3X^{80}Y^{57} - 2X^{82}Y^{57} - 8X^{83}Y^{57} \\
& + 14X^{84}Y^{57} - 4X^{85}Y^{57} + 2X^{70}Y^{58} + 4X^{71}Y^{58} - 21X^{72}Y^{58} + 13X^{73}Y^{58} \\
& - 2X^{74}Y^{58} + 3X^{76}Y^{58} + 4X^{77}Y^{58} - 52X^{78}Y^{58} + 47X^{79}Y^{58} - 8X^{80}Y^{58} \\
& + X^{81}Y^{58} - 12X^{83}Y^{58} + 12X^{84}Y^{58} - X^{85}Y^{58} + 5X^{72}Y^{59} - 10X^{73}Y^{59} \\
& + 3X^{74}Y^{59} + X^{76}Y^{59} + 18X^{77}Y^{59} - 56X^{78}Y^{59} + 29X^{79}Y^{59} + 2X^{80}Y^{59} \\
& - X^{81}Y^{59} + 2X^{82}Y^{59} + X^{83}Y^{59} - 29X^{84}Y^{59} + 27X^{85}Y^{59} - 4X^{86}Y^{59} \\
& + 3X^{72}Y^{60} - 5X^{73}Y^{60} + X^{74}Y^{60} + 7X^{77}Y^{60} - 36X^{79}Y^{60} + 27X^{80}Y^{60} \\
& - 5X^{81}Y^{60} + 2X^{82}Y^{60} + 8X^{83}Y^{60} - 33X^{84}Y^{60} + 17X^{85}Y^{60} - X^{77}Y^{61}
\end{aligned}$$

$$\begin{aligned}
& + 19X^{78}Y^{61} - 36X^{79}Y^{61} + 12X^{80}Y^{61} + X^{81}Y^{61} + 6X^{83}Y^{61} + 3X^{84}Y^{61} \\
& - 37X^{85}Y^{61} + 23X^{86}Y^{61} - 2X^{87}Y^{61} + X^{89}Y^{61} - 5X^{90}Y^{61} + 3X^{91}Y^{61} \\
& + 5X^{78}Y^{62} + 6X^{79}Y^{62} - 20X^{80}Y^{62} + 7X^{81}Y^{62} + 2X^{83}Y^{62} + 14X^{84}Y^{62} \\
& - 40X^{85}Y^{62} + 20X^{86}Y^{62} - 3X^{87}Y^{62} - X^{91}Y^{62} + 6X^{79}Y^{63} - 7X^{80}Y^{63} \\
& + X^{81}Y^{63} - 2X^{83}Y^{63} + 11X^{84}Y^{63} + 2X^{85}Y^{63} - 28X^{86}Y^{63} + 11X^{87}Y^{63} \\
& + 7X^{90}Y^{63} - 14X^{91}Y^{63} + 5X^{92}Y^{63} + 3X^{80}Y^{64} - 2X^{81}Y^{64} - X^{83}Y^{64} \\
& + 8X^{84}Y^{64} + 14X^{85}Y^{64} - 38X^{86}Y^{64} + 18X^{87}Y^{64} - 3X^{88}Y^{64} + 2X^{90}Y^{64} \\
& - 3X^{91}Y^{64} - X^{92}Y^{64} + 2X^{79}Y^{65} - X^{80}Y^{65} - 2X^{84}Y^{65} + 6X^{85}Y^{65} \\
& + 8X^{86}Y^{65} - 14X^{87}Y^{65} + 2X^{88}Y^{65} + X^{90}Y^{65} + 15X^{91}Y^{65} - 24X^{92}Y^{65} \\
& + 7X^{93}Y^{65} - 3X^{84}Y^{66} + 8X^{85}Y^{66} + 13X^{86}Y^{66} - 22X^{87}Y^{66} + 6X^{88}Y^{66} \\
& + X^{90}Y^{66} + 8X^{91}Y^{66} - 7X^{92}Y^{66} - X^{93}Y^{66} + X^{85}Y^{67} + 2X^{87}Y^{67} \\
& - X^{88}Y^{67} - 3X^{90}Y^{67} + 4X^{91}Y^{67} + 21X^{92}Y^{67} - 26X^{93}Y^{67} + 6X^{94}Y^{67} \\
& - 3X^{85}Y^{68} + 3X^{86}Y^{68} + 5X^{87}Y^{68} - 3X^{88}Y^{68} - X^{90}Y^{68} + 11X^{92}Y^{68} \\
& - 5X^{93}Y^{68} - X^{94}Y^{68} + 2X^{97}Y^{68} - X^{98}Y^{68} - 4X^{91}Y^{69} + 19X^{93}Y^{69} \\
& - 11X^{94}Y^{69} + 3X^{98}Y^{69} - 2X^{99}Y^{69} - 2X^{91}Y^{70} + 8X^{93}Y^{70} - 3X^{94}Y^{70} \\
& - 3X^{97}Y^{70} + 7X^{98}Y^{70} - X^{99}Y^{70} - X^{100}Y^{70} - 3X^{92}Y^{71} + 4X^{93}Y^{71} \\
& + X^{94}Y^{71} - 5X^{98}Y^{71} + 9X^{99}Y^{71} - 2X^{100}Y^{71} - X^{92}Y^{72} + 2X^{94}Y^{72} \\
& - 5X^{98}Y^{72} + 8X^{99}Y^{72} - X^{100}Y^{72} - 3X^{93}Y^{73} + 3X^{94}Y^{73} - X^{95}Y^{73} \\
& + X^{98}Y^{73} - 4X^{99}Y^{73} + 4X^{100}Y^{73} - 9X^{99}Y^{74} + 10X^{100}Y^{74} - X^{101}Y^{74} \\
& - 2X^{99}Y^{75} + X^{100}Y^{75} + 2X^{99}Y^{76} - 6X^{100}Y^{76} + 3X^{101}Y^{76} - 3X^{105}Y^{76} \\
& + 3X^{106}Y^{76} - X^{107}Y^{76} - 2X^{100}Y^{77} + X^{101}Y^{77} - 2X^{106}Y^{77} + X^{107}Y^{77} \\
& + 2X^{105}Y^{78} - 6X^{106}Y^{78} + 3X^{107}Y^{78} + X^{106}Y^{79} - 2X^{107}Y^{79} + 3X^{106}Y^{80} \\
& - 4X^{107}Y^{80} + X^{113}Y^{85}.
\end{aligned}$$

$\zeta_{H \times H \times M_3}^{\triangleleft}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{H \times H \times M_3, p}^{\triangleleft}(s) \Big|_{p \rightarrow p^{-1}} = p^{45-19s} \zeta_{H \times H \times M_3, p}^{\triangleleft}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{H \times H \times M_3}^{\triangleleft}(s)$ is 6, with a simple pole at $s = 6$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(s-5)\zeta_p(3s-6)^3\zeta_p(4s-6) \\ & \times \zeta_p(5s-7)\zeta_p(6s-7)\zeta_p(7s-8)\zeta_p(8s-8)\zeta_p(8s-14)\zeta_p(9s-9)\zeta_p(9s-14) \\ & \times \zeta_p(10s-15)\zeta_p(11s-16)\zeta_p(12s-21)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s}) \\ & \times W_4(p,p^{-s})W_5(p,p^{-s})W_6(p,p^{-s})W_7(p,p^{-s}) \end{aligned}$$

where

$$\begin{aligned} W_1(X, Y) &= 1 + 2X^{34}Y^{20}, \\ W_2(X, Y) &= 2 + X^{14}Y^9, \\ W_3(X, Y) &= 1 - X^{15}Y^{10}, \\ W_4(X, Y) &= -1 - X^{21}Y^{16}, \\ W_5(X, Y) &= -1 + 3X^7Y^6, \\ W_6(X, Y) &= 3 - X^{16}Y^{15}, \\ W_7(X, Y) &= -1 + X^6Y^9. \end{aligned}$$

The ghost is unfriendly.

6 Natural boundary

$\zeta_{H \times H \times M_3}^{\triangleleft}(s)$ has a natural boundary at $\Re(s) = 17/10$, and is of type I.