The zeta function of $H(\mathcal{O}_{K_3})$ counting ideals

1 Presentation

Let K_3 be a cubic number field, and $\mathcal{O}_{K_3}$ its ring of integers. $H(\mathcal{O}_{K_3})$, the Heisenberg Lie ring over $\mathcal{O}_{K_3}$, has presentation

$$\langle x, y, z \mid [x, y] = z \rangle .$$

as a $\mathcal{O}_{K_3}$-Lie ring. Given any fixed cubic Galois number field K_3, and an integral basis for $\mathcal{O}_{K_3}$, one can construct a presentation for $\mathcal{O}_{K_3}$ as a $\mathbb{Z}$-Lie ring. However, I know of no way to write down an integral basis for an arbitrary cubic number field, so I have left the presentation in the form above.

$H(\mathcal{O}_{K_3})$ has nilpotency class 2, and rank 9 as a $\mathbb{Z}$-Lie ring.

2 The local zeta function

The local zeta functions depend on the behaviour of the prime p in the ring of integers of K_3. The local zeta functions for inert and totally ramified primes were first calculated by Grunewald, Segal & Smith. Taylor contributed the case for totally split primes, and the remaining two cases, partially ramified and partially split, were calculated by Woodward. These last two cases only arise if K_3 is *not* a normal extension of $\mathbb{Q}$. We have that

$$\begin{aligned}\zeta^{\triangleleft}_{H(\mathcal{O}_{K_3}),p}(s) = {}& \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(s-5)\zeta_p(5s-7) \\ & \times \zeta_p(7s-8)\zeta_p(8s-14)\zeta_{K_3,p}(3s-6)W_p(p,p^{-s}),\end{aligned}$$

where $\zeta_{K_3,p}(s)$ is the p-local factor of the Dedekind zeta function, defined by

$$\zeta_{K_3,p}(s) = \prod_{\mathfrak{p}|p} \frac{1}{1-(N_{K_3/\mathbb{Q}}(\mathfrak{p}))^{-s}},$$

where the product is over all prime ideals $\mathfrak{p}$ dividing p.

$W_p(p,p^{-s})$ depends on the behaviour of p in $\mathcal{O}_{K_3}$:

- If p is inert,

$$W_p(X,Y) = (1-X^7Y^5)(1+X^6Y^7+X^7Y^7+X^{12}Y^8+X^{13}Y^8+X^{19}Y^{15}).$$

- If p ramifies totally (i.e. $(p) = \mathfrak{p}^3$ for some prime ideal $\mathfrak{p}$),

$$W_p(X,Y) = 1 - X^{14}Y^{10}.$$

- If p splits totally,

$$W_p(X,Y) = 1 - 3X^6Y^5 + 2X^7Y^5 + X^6Y^7 - 2X^7Y^7 + X^{12}Y^8 - 2X^{13}Y^8 + 2X^{13}Y^{12} - X^{14}Y^{12} + 2X^{19}Y^{13} - X^{20}Y^{13} - 2X^{19}Y^{15} + 3X^{20}Y^{15} - X^{26}Y^{20}.$$

- If p ramifies partially (i.e. $(p) = \mathfrak{p}^2\mathfrak{q}$ for prime ideals $\mathfrak{p} \neq \mathfrak{q}$),

$$W_p(X,Y) = 1 - X^6Y^5 + X^7Y^5 - X^7Y^7 - X^{13}Y^8 + X^{13}Y^{10} - X^{14}Y^{10} + X^{20}Y^{15}.$$

- If p splits partially (i.e. $(p) = \mathfrak{p}\mathfrak{q}$ for prime ideals $\mathfrak{p}$, $\mathfrak{q}$):

$$W_p(X,Y) = 1 + X^6Y^5 - X^6Y^7 - X^{12}Y^8 - X^{14}Y^{12} - X^{20}Y^{13} + X^{20}Y^{15} + X^{26}Y^{20}.$$

3 Functional equation

For p split or inert, the local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{H(\mathcal{O}_{K_3}),p}(s)|_{p\to p^{-1}} = -p^{36-15s}\zeta^{\triangleleft}_{H(\mathcal{O}_{K_3}),p}(s).$$

The p ramified cases also satisfy functional equations: if p is partially ramified, then

$$\zeta^{\triangleleft}_{H(\mathcal{O}_{K_3}),p}(s)|_{p\to p^{-1}} = -p^{36-17s}\zeta^{\triangleleft}_{H(\mathcal{O}_{K_3}),p}(s),$$

and if p is totally ramified then

$$\zeta^{\triangleleft}_{H(\mathcal{O}_{K_3}),p}(s)|_{p\to p^{-1}} = -p^{36-19s}\zeta^{\triangleleft}_{H(\mathcal{O}_{K_3}),p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{H(\mathcal{O}_{K_3})}(s)$ is 6, with a simple pole at $s = 6$.

5 Ghost zeta function

The ghost zeta function is unknown, since the local zeta functions vary with the prime.

6 Natural boundary

The natural boundary is believed to be at $\Re(s) = 13/8$, but since the local zeta functions vary with the prime, this has not been confirmed.