The zeta function of H^3 counting ideals

1 Presentation

H^3 has presentation

$$\langle x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \mid [x_1, y_1] = z_1, [x_2, y_2] = z_2, [x_3, y_3] = z_3 \rangle .$$

H^3 has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Gareth Taylor. It is

$$\begin{aligned}\zeta^{\triangleleft}_{H^3,p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(s-5)\zeta_p(3s-6)^3 \\ & \times \zeta_p(5s-7)\zeta_p(7s-8)\zeta_p(8s-14)W(p,p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned}& 1 - 3X^6Y^5 + 2X^7Y^5 + X^6Y^7 - 2X^7Y^7 + X^{12}Y^8 - 2X^{13}Y^8 + 2X^{13}Y^{12} \\ & - X^{14}Y^{12} + 2X^{19}Y^{13} - X^{20}Y^{13} - 2X^{19}Y^{15} + 3X^{20}Y^{15} - X^{26}Y^{20}.\end{aligned}$$

$\zeta^{\triangleleft}_{H^3}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\left.\zeta^{\triangleleft}_{H^3,p}(s)\right|_{p\to p^{-1}} = -p^{36-15s}\zeta^{\triangleleft}_{H^3,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{H^3}(s)$ is 6, with a simple pole at $s = 6$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}& \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(s-5)\zeta_p(3s-6)^3\zeta_p(5s-7) \\ & \times \zeta_p(7s-8)\zeta_p(8s-14)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})\end{aligned}$$

where

$$W_1(X,Y) = 1 - 2X^{13}Y^8,$$
$$W_2(X,Y) = -2 - X^7Y^5,$$
$$W_3(X,Y) = -1 - X^6Y^7.$$

The ghost is unfriendly.

6 Natural boundary

$\zeta^{\triangleleft}_{H^3}(s)$ has a natural boundary at $\Re(s) = 13/8$, and is of type I.