The zeta function of G_5 counting ideals

1 Presentation

G_5 has presentation

$$\left\langle z, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \;\middle|\; \begin{array}{c} [z, x_1] = y_1, [z, x_2] = y_2, \\ [z, x_3] = y_3, [z, x_4] = y_4 \end{array} \right\rangle .$$

G_5 has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Dermot Grenham. It is

$$\begin{aligned} \zeta^{\triangleleft}_{G_5,p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(3s-8)\zeta_p(5s-14) \\ & \times \zeta_p(7s-18)\zeta_p(9s-20)W(p, p^{-s}) \end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned} & 1 + X^5Y^3 + X^6Y^3 + X^7Y^3 + X^{10}Y^5 + X^{11}Y^5 + 2X^{12}Y^5 + X^{13}Y^5 + X^{15}Y^7 \\ & + X^{16}Y^7 + X^{17}Y^7 + X^{17}Y^8 + X^{18}Y^8 + X^{19}Y^8 + X^{21}Y^{10} + 2X^{22}Y^{10} \\ & + X^{23}Y^{10} + X^{24}Y^{10} + X^{27}Y^{12} + X^{28}Y^{12} + X^{29}Y^{12} + X^{34}Y^{15}. \end{aligned}$$

$\zeta^{\triangleleft}_{G_5}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{G_5,p}(s)\Big|_{p \to p^{-1}} = -p^{36-14s}\zeta^{\triangleleft}_{G_5,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{G_5}(s)$ is 5, with a simple pole at $s = 5$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(3s-8)\zeta_p(5s-14)\zeta_p(7s-18) \\ & \times \zeta_p(9s-20)W_1(p, p^{-s})W_2(p, p^{-s})W_3(p, p^{-s}) \end{aligned}$$

where

$$
\begin{aligned}
W_1(X,Y) &= 1 + X^{13}Y^5, \\
W_2(X,Y) &= 1 + X^{16}Y^7, \\
W_3(X,Y) &= 1 + X^5Y^3.
\end{aligned}
$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{G_5}(s)$ has a natural boundary at $\Re(s) = 13/5$, and is of type III.