The zeta function of G_4 counting ideals

1 Presentation

G_4 has presentation

$$\langle z, x_1, x_2, x_3, y_1, y_2, y_3 \mid [z, x_1] = y_1, [z, x_2] = y_2, [z, x_3] = y_3 \rangle .$$

G_4 has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Dermot Grenham. It is

$$\zeta^{\triangleleft}_{G_4,p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-6)\zeta_p(5s-10)\zeta_p(7s-12) \times W(p, p^{-s})$$

where $W(X, Y)$ is

$$1 + X^4Y^3 + X^5Y^3 + X^8Y^5 + X^9Y^5 + X^{13}Y^8.$$

$\zeta^{\triangleleft}_{G_4}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{G_4,p}(s)\Big|_{p\to p^{-1}} = -p^{21-11s}\zeta^{\triangleleft}_{G_4,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{G_4}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-6)\zeta_p(5s-10)\zeta_p(7s-12)W_1(p, p^{-s}) \times W_2(p, p^{-s})$$

where

$$W_1(X,Y) = 1 + X^9Y^5,$$
$$W_2(X,Y) = 1 + X^4Y^3.$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{G_4}(s)$ has a natural boundary at $\Re(s) = 9/5$, and is of type III.