

The zeta function of $F_{3,2} \times \mathbb{Z}$ counting ideals

1 Presentation

$F_{3,2} \times \mathbb{Z}$ has presentation

$$\langle x_1, x_2, y, a, z_1, z_2 \mid [x_1, x_2] = y, [x_1, y] = z_1, [x_2, y] = z_2 \rangle.$$

$F_{3,2} \times \mathbb{Z}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta_{F_{3,2} \times \mathbb{Z}, p}^{\triangleleft}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(5s-6) \\ \times \zeta_p(7s-8)W(p, p^{-s})$$

where $W(X, Y)$ is

$$1 + X^3Y^4 - X^3Y^5 - X^6Y^7 - X^8Y^9 - X^{11}Y^{11} + X^{11}Y^{12} + X^{14}Y^{16}.$$

$\zeta_{F_{3,2} \times \mathbb{Z}}^{\triangleleft}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{F_{3,2} \times \mathbb{Z}, p}^{\triangleleft}(s) \Big|_{p \rightarrow p^{-1}} = p^{15-11s} \zeta_{F_{3,2} \times \mathbb{Z}, p}^{\triangleleft}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{F_{3,2} \times \mathbb{Z}}^{\triangleleft}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(5s-6)\zeta_p(7s-8) \\ \times W_1(p, p^{-s})W_2(p, p^{-s})$$

where

$$\begin{aligned}W_1(X, Y) &= 1 - X^{11}Y^{11}, \\W_2(X, Y) &= -1 + X^3Y^5.\end{aligned}$$

The ghost is friendly.

6 Natural boundary

$\zeta_{F_{3,2} \times \mathbb{Z}}^{\triangleleft}(s)$ has a natural boundary at $\Re(s) = 1$, and is of type II.