The zeta function of $F_{3,2}$ counting all subrings

1 Presentation

$F_{3,2}$ has presentation

$$\langle x_1, x_2, y_1, z_1, z_2 \mid [x_1, x_2] = y_1, [x_1, y_1] = z_1, [x_2, y_1] = z_2 \rangle .$$

$F_{3,2}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta_{F_{3,2},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(2s-3)\zeta_p(2s-4)\zeta_p(3s-6)\zeta_p(4s-8)\zeta_p(5s-8) \\ & \times \zeta_p(5s-9)W(p, p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned} & 1 + X^2Y^2 + X^3Y^2 - X^3Y^3 + X^4Y^3 + 2X^5Y^3 - 2X^5Y^4 + 2X^7Y^4 - 2X^7Y^5 \\ & - 2X^8Y^5 - X^9Y^5 - X^{10}Y^6 - X^{11}Y^6 - X^{10}Y^7 - X^{13}Y^7 - 2X^{12}Y^8 - X^{13}Y^8 \\ & - X^{14}Y^8 - X^{15}Y^8 + X^{13}Y^9 - X^{16}Y^9 + X^{14}Y^{10} + X^{15}Y^{10} + X^{16}Y^{10} \\ & + 2X^{17}Y^{10} + X^{16}Y^{11} + X^{19}Y^{11} + X^{18}Y^{12} + X^{19}Y^{12} + X^{20}Y^{13} + 2X^{21}Y^{13} \\ & + 2X^{22}Y^{13} - 2X^{22}Y^{14} + 2X^{24}Y^{14} - 2X^{24}Y^{15} - X^{25}Y^{15} + X^{26}Y^{15} \\ & - X^{26}Y^{16} - X^{27}Y^{16} - X^{29}Y^{18}. \end{aligned}$$

$\zeta_{F_{3,2}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{F_{3,2},p}(s)\big|_{p\to p^{-1}} = -p^{10-5s}\zeta_{F_{3,2},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{F_{3,2}}(s)$ is $5/2$, with a simple pole at $s = 5/2$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(2s-3)\zeta_p(2s-4)\zeta_p(3s-6)\zeta_p(4s-8)\zeta_p(5s-8)\zeta_p(5s-9) \\ \times W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})$$

where

$$\begin{aligned} W_1(X,Y) &= 1 - X^{15}Y^8, \\ W_2(X,Y) &= -1 + X^{11}Y^7, \\ W_3(X,Y) &= 1 - XY - X^3Y^3. \end{aligned}$$

The ghost is unfriendly.

6 Natural boundary

$\zeta_{F_{3,2}}(s)$ has a natural boundary at $\Re(s) = 15/8$, and is of type III.

7 Notes

This calculation was inspired by Gareth Taylor's calculation of $\zeta_{F_{2,3},p}(s)$. The conditions under both cone integrals are very similar, and Luke Woodward was able to mimic Gareth Taylor's calculation to achieve this result.

The global zeta function is also the third discovered, and the first at class 3, where the abscissa of convergence is strictly greater than the rank of the abelianisation.