The zeta function of $F_{3,2}$ counting ideals

1 Presentation

$F_{3,2}$ has presentation

$$\langle x_1, x_2, y_1, z_1, z_2 \mid [x_1, x_2] = y_1, [x_1, y_1] = z_1, [x_2, y_1] = z_2 \rangle .$$

$F_{3,2}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{F_{3,2},p}(s) = &\zeta_p(s)\zeta_p(s-1)\zeta_p(3s-2)\zeta_p(4s-3)\zeta_p(5s-4)^2\zeta_p(7s-6)\\ &\times W(p, p^{-s})\end{aligned}$$

where $W(X, Y)$ is

$$1 + X^2Y^4 - X^2Y^5 - X^4Y^7 - X^6Y^9 - X^8Y^{11} + X^8Y^{12} + X^{10}Y^{16}.$$

$\zeta^{\triangleleft}_{F_{3,2}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{F_{3,2},p}(s)\Big|_{p\to p^{-1}} = -p^{10-10s}\zeta^{\triangleleft}_{F_{3,2},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{F_{3,2}}(s)$ is 2, with a simple pole at $s = 2$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(3s-2)\zeta_p(4s-3)\zeta_p(5s-4)^2\zeta_p(7s-6)W_1(p, p^{-s})W_2(p, p^{-s})$$

where

$$\begin{aligned}W_1(X, Y) &= 1 - X^8Y^{11},\\ W_2(X, Y) &= -1 + X^2Y^5.\end{aligned}$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{F_{3,2}}(s)$ has a natural boundary at $\Re(s) = 8/11$, and is of type III.