The zeta function of $F_{2,3}$ counting ideals

1 Presentation

$F_{2,3}$ has presentation

$$\langle x_1, x_2, x_3, y_1, y_2, y_3 \mid [x_1, x_2] = y_1, [x_1, x_3] = y_2, [x_2, x_3] = y_3 \rangle .$$

$F_{2,3}$ has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Grunewald, Segal & Smith. It is

$$\zeta^{\triangleleft}_{F_{2,3},p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-5)\zeta_p(5s-8)\zeta_p(6s-9)W(p, p^{-s})$$

where $W(X, Y)$ is

$$1 + X^3Y^3 + X^4Y^3 + X^6Y^5 + X^7Y^5 + X^{10}Y^8.$$

$\zeta^{\triangleleft}_{F_{2,3}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{F_{2,3},p}(s)\Big|_{p\to p^{-1}} = p^{15-9s}\zeta^{\triangleleft}_{F_{2,3},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{F_{2,3}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-5)\zeta_p(5s-8)\zeta_p(6s-9)W_1(p, p^{-s})W_2(p, p^{-s})$$

where

$$W_1(X, Y) = 1 + X^7Y^5,$$
$$W_2(X, Y) = 1 + X^3Y^3.$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\lhd}_{F_{2,3}}(s)$ has a natural boundary at $\Re(s) = 7/5$, and is of type III.