The zeta function of $\mathfrak{g}_{37C}$ counting ideals

1 Presentation

$\mathfrak{g}_{37C}$ has presentation

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \;\middle|\; \begin{array}{c} [x_1, x_2] = x_5, [x_2, x_3] = x_6, \\ [x_2, x_4] = x_7, [x_3, x_4] = x_5 \end{array} \right\rangle.$$

$\mathfrak{g}_{37C}$ has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta^{\triangleleft}_{\mathfrak{g}_{37C},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-5)^2\zeta_p(5s-6)\zeta_p(5s-8) \\ & \times \zeta_p(6s-10)\zeta_p(7s-12)W(p, p^{-s}) \end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned} & 1 + X^4Y^3 - X^5Y^5 + X^8Y^5 - X^8Y^6 - X^9Y^6 - X^{10}Y^8 - X^{12}Y^8 - X^{13}Y^9 \\ & + X^{13}Y^{10} - 2X^{14}Y^{10} + X^{14}Y^{11} + X^{15}Y^{11} - X^{16}Y^{11} - X^{17}Y^{11} + 2X^{17}Y^{12} \\ & - X^{18}Y^{12} + X^{18}Y^{13} + X^{19}Y^{14} + X^{21}Y^{14} + X^{22}Y^{16} + X^{23}Y^{16} - X^{23}Y^{17} \\ & + X^{26}Y^{17} - X^{27}Y^{19} - X^{31}Y^{22}. \end{aligned}$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{37C}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{37C},p}(s)\Big|_{p \to p^{-1}} = -p^{21-11s}\zeta^{\triangleleft}_{\mathfrak{g}_{37C},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{g}_{37C}}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-5)^2\zeta_p(5s-6)\zeta_p(5s-8)\zeta_p(6s-10)$$
$$\times \zeta_p(7s-12)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})$$

where

$$\begin{aligned} W_1(X,Y) &= 1 + X^8Y^5, \\ W_2(X,Y) &= 1 - X^9Y^6 + X^{18}Y^{12}, \\ W_3(X,Y) &= 1 - X^5Y^5. \end{aligned}$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{g}_{37C}}(s)$ has a natural boundary at $\Re(s) = 8/5$, and is of type III.

7 Notes

This ideal zeta function is identical to that of T_4, though the Lie rings themselves are non-isomorphic.