The zeta function of $\mathfrak{g}_{257K}$ counting ideals

1 Presentation

$\mathfrak{g}_{257K}$ has presentation

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \;\middle|\; \begin{array}{c} [x_1, x_2] = x_5, [x_1, x_5] = x_6, \\ [x_2, x_5] = x_7, [x_3, x_4] = x_6 \end{array} \right\rangle .$$

$\mathfrak{g}_{257K}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta^{\triangleleft}_{\mathfrak{g}_{257K},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-5) \\ & \times \zeta_p(6s-5)\zeta_p(7s-6)\zeta_p(7s-8)\zeta_p(9s-10)W(p, p^{-s}) \end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned} & 1 - X^4Y^5 - X^5Y^7 - X^8Y^9 - X^8Y^{10} + X^8Y^{11} - X^{10}Y^{11} + X^9Y^{12} + X^{12}Y^{13} \\ & - X^{13}Y^{13} + X^{13}Y^{14} + 2X^{13}Y^{15} - X^{14}Y^{15} - X^{13}Y^{16} + 2X^{14}Y^{16} + X^{14}Y^{17} \\ & - X^{14}Y^{18} + X^{15}Y^{18} + X^{18}Y^{19} - X^{17}Y^{20} + X^{19}Y^{20} - X^{19}Y^{21} - X^{19}Y^{22} \\ & - X^{22}Y^{24} - X^{23}Y^{26} + X^{27}Y^{31}. \end{aligned}$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{257K}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{257K},p}(s)\Big|_{p \to p^{-1}} = -p^{21-14s}\zeta^{\triangleleft}_{\mathfrak{g}_{257K},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{g}_{257K}}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(6s-5)$$
$$\times\,\zeta_p(7s-6)\zeta_p(7s-8)\zeta_p(9s-10)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})W_4(p,p^{-s})$$

where

$$W_1(X,Y) = 1 - X^{13}Y^{13},$$
$$W_2(X,Y) = -1 + X^{6}Y^{7},$$
$$W_3(X,Y) = 1 - X^{3}Y^{4},$$
$$W_4(X,Y) = -1 + X^{5}Y^{7}.$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{g}_{257K}}(s)$ has a natural boundary at $\Re(s) = 1$, and is of type II.

7 Notes

The Lie ring $\mathfrak{g}_{257K}$ is isomorphic to the central amalgamation of the free class-3-nilpotent 2-generator Lie ring $F_{3,2}$ with the Heisenberg Lie ring H.