The zeta function of $\mathfrak{g}_{1457B}$ counting ideals

1 Presentation

$\mathfrak{g}_{1457B}$ has presentation

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \;\middle|\; \begin{array}{c} [x_1, x_2] = x_5, [x_1, x_5] = x_6, [x_1, x_6] = x_7, \\ [x_2, x_5] = x_7, [x_3, x_4] = x_7 \end{array} \right\rangle.$$

$\mathfrak{g}_{1457B}$ has nilpotency class 4.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta^{\triangleleft}_{\mathfrak{g}_{1457B},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-5) \\ & \times \zeta_p(7s-4)\zeta_p(9s-6)\zeta_p(10s-9)\zeta_p(11s-10)\zeta_p(12s-10) \\ & \times \zeta_p(16s-11)W(p,p^{-s}) \end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned} & 1 - X^4Y^5 - X^4Y^8 + X^5Y^8 - X^5Y^9 - X^8Y^{10} + X^8Y^{11} - 2X^9Y^{11} + X^8Y^{12} \\ & - X^9Y^{12} - X^{10}Y^{12} + 2X^9Y^{13} - 2X^{10}Y^{13} + X^{10}Y^{14} + 2X^{13}Y^{15} - X^{14}Y^{15} \\ & - X^{10}Y^{16} - X^{13}Y^{16} + 2X^{14}Y^{16} + X^{10}Y^{17} - X^{11}Y^{17} + X^{14}Y^{17} + X^{13}Y^{18} \\ & - X^{14}Y^{18} + 3X^{14}Y^{19} - 2X^{15}Y^{19} + X^{15}Y^{20} + 3X^{15}Y^{21} - X^{16}Y^{21} \\ & + X^{18}Y^{21} - X^{15}Y^{22} + X^{16}Y^{22} + X^{19}Y^{22} - X^{17}Y^{23} - X^{18}Y^{23} + 3X^{19}Y^{23} \\ & - X^{18}Y^{24} - 2X^{19}Y^{24} + 2X^{20}Y^{24} + X^{15}Y^{25} - X^{18}Y^{25} + X^{19}Y^{26} - X^{20}Y^{26} \\ & - X^{23}Y^{26} + X^{20}Y^{27} - X^{22}Y^{27} - X^{19}Y^{28} + X^{20}Y^{28} + X^{21}Y^{28} + X^{22}Y^{28} \\ & - X^{23}Y^{28} - X^{24}Y^{28} - X^{19}Y^{29} + X^{21}Y^{29} - X^{20}Y^{30} - X^{23}Y^{30} - 3X^{23}Y^{31} \\ & + X^{25}Y^{31} + 2X^{23}Y^{32} - 3X^{24}Y^{32} - X^{25}Y^{32} - X^{25}Y^{33} - X^{27}Y^{33} + X^{24}Y^{34} \\ & - X^{25}Y^{34} + 2X^{27}Y^{34} - 2X^{28}Y^{34} - X^{24}Y^{35} + X^{27}Y^{35} + X^{28}Y^{35} - X^{29}Y^{35} \\ & - X^{25}Y^{36} + 3X^{28}Y^{36} - 2X^{29}Y^{36} - X^{25}Y^{37} - 2X^{28}Y^{37} + 2X^{29}Y^{37} \\ & - X^{29}Y^{38} + X^{32}Y^{38} + 2X^{28}Y^{39} - 2X^{29}Y^{39} - X^{30}Y^{39} - X^{32}Y^{39} + X^{33}Y^{39} \\ & + 4X^{29}Y^{40} - 3X^{30}Y^{40} + X^{29}Y^{41} + X^{30}Y^{41} - X^{31}Y^{41} + X^{32}Y^{41} - X^{33}Y^{41} \\ & + X^{30}Y^{42} - X^{32}Y^{42} + 4X^{33}Y^{42} - X^{34}Y^{42} + 2X^{34}Y^{43} - X^{35}Y^{43} \\ & - 2X^{33}Y^{44} + 3X^{34}Y^{44} - 2X^{34}Y^{45} + 2X^{35}Y^{45} + X^{34}Y^{46} - X^{37}Y^{46} \\ & + X^{38}Y^{46} + 2X^{34}Y^{47} - X^{35}Y^{47} - 2X^{38}Y^{47} + X^{39}Y^{47} - X^{34}Y^{48} \end{aligned}$$

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\begin{aligned}
&+ 2X^{35}Y^{48} + X^{38}Y^{48} - X^{39}Y^{48} + X^{38}Y^{49} - 2X^{38}Y^{50} + 2X^{39}Y^{50} \\
&- X^{38}Y^{51} - X^{39}Y^{51} + X^{40}Y^{51} - 2X^{39}Y^{52} + X^{40}Y^{52} - X^{40}Y^{53} - X^{43}Y^{54} \\
&+ X^{43}Y^{55} - X^{44}Y^{55} - X^{44}Y^{58} + X^{48}Y^{63}.
\end{aligned}
$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{1457B}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies no functional equation.

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{g}_{1457B}}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

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\begin{aligned}
&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(7s-4) \\
&\times \zeta_p(9s-6)\zeta_p(10s-9)\zeta_p(11s-10)\zeta_p(12s-10)\zeta_p(16s-11)W_1(p,p^{-s}) \\
&\times W_2(p,p^{-s})W_3(p,p^{-s})W_4(p,p^{-s})W_5(p,p^{-s})W_6(p,p^{-s})
\end{aligned}
$$

where

$$
\begin{aligned}
W_1(X,Y) &= 1 - X^{14}Y^{15}, \\
W_2(X,Y) &= -1 - X^{9}Y^{11}, \\
W_3(X,Y) &= -1 + X^{10}Y^{13}, \\
W_4(X,Y) &= 1 + X^{6}Y^{8}, \\
W_5(X,Y) &= 1 - X^{5}Y^{8}, \\
W_6(X,Y) &= -1 + X^{4}Y^{8}.
\end{aligned}
$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{g}_{1457B}}(s)$ has a natural boundary at $\Re(s) = 14/15$, and is of type III.

7 Notes

The Lie ring $\mathfrak{g}_{1457B}$ is isomorphic to the central amalgamation of the maximal class Lie ring Fil_4 with the Heisenberg Lie ring H.