

# The zeta function of $\mathfrak{g}_{137C}$ counting ideals

## 1 Presentation

$\mathfrak{g}_{137C}$  has presentation

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \left| \begin{array}{l} [x_1, x_2] = x_5, [x_1, x_4] = x_6, [x_1, x_6] = x_7, \\ [x_2, x_3] = x_6, [x_3, x_5] = -x_7 \end{array} \right. \right\rangle.$$

$\mathfrak{g}_{137C}$  has nilpotency class 3.

## 2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta_{\mathfrak{g}_{137C}, p}^{\triangleleft}(s) &= \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(5s-5)\zeta_p(6s-9) \\ &\quad \times \zeta_p(7s-4)\zeta_p(9s-6)\zeta_p(11s-10)\zeta_p(12s-10)\zeta_p(16s-11) \\ &\quad \times W(p, p^{-s}) \end{aligned}$$

where  $W(X, Y)$  is

$$\begin{aligned} &1 - X^4Y^8 + X^5Y^8 - X^9Y^8 - X^5Y^9 - X^9Y^{11} - X^{10}Y^{12} + X^9Y^{13} - X^{10}Y^{13} \\ &+ X^{13}Y^{15} - X^{14}Y^{15} - X^{10}Y^{16} + X^{14}Y^{16} - X^{15}Y^{16} + X^{10}Y^{17} - X^{11}Y^{17} \\ &+ X^{15}Y^{17} + X^{14}Y^{19} - X^{15}Y^{19} + X^{19}Y^{19} + X^{15}Y^{20} + X^{19}Y^{20} + X^{14}Y^{21} \\ &+ X^{15}Y^{21} - X^{16}Y^{21} - X^{15}Y^{22} + X^{16}Y^{22} + X^{18}Y^{23} + X^{19}Y^{23} - X^{20}Y^{23} \\ &- X^{18}Y^{24} - X^{19}Y^{24} + 3X^{20}Y^{24} + X^{15}Y^{25} - X^{23}Y^{26} + X^{24}Y^{26} + X^{19}Y^{27} \\ &- X^{19}Y^{28} + X^{20}Y^{28} + X^{21}Y^{28} - X^{23}Y^{28} - X^{24}Y^{28} + X^{25}Y^{28} - X^{25}Y^{29} \\ &- X^{20}Y^{30} + X^{21}Y^{30} - X^{29}Y^{31} - 3X^{24}Y^{32} + X^{25}Y^{32} + X^{26}Y^{32} + X^{24}Y^{33} \\ &- X^{25}Y^{33} - X^{26}Y^{33} - X^{28}Y^{34} + X^{29}Y^{34} + X^{28}Y^{35} - X^{29}Y^{35} - X^{30}Y^{35} \\ &- X^{25}Y^{36} - X^{29}Y^{36} - X^{25}Y^{37} + X^{29}Y^{37} - X^{30}Y^{37} - X^{29}Y^{39} + X^{33}Y^{39} \\ &- X^{34}Y^{39} + X^{29}Y^{40} - X^{30}Y^{40} + X^{34}Y^{40} + X^{30}Y^{41} - X^{31}Y^{41} + X^{34}Y^{43} \\ &- X^{35}Y^{43} + X^{34}Y^{44} + X^{35}Y^{45} + X^{39}Y^{47} + X^{35}Y^{48} - X^{39}Y^{48} + X^{40}Y^{48} \\ &- X^{44}Y^{56}. \end{aligned}$$

$\zeta_{\mathfrak{g}_{137C}}^{\triangleleft}(s)$  is uniform.

### 3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{\mathfrak{g}_{137C},p}^{\triangleleft}(s)|_{p \rightarrow p^{-1}} = -p^{21-17s} \zeta_{\mathfrak{g}_{137C},p}^{\triangleleft}(s).$$

### 4 Abscissa of convergence and order of pole

The abscissa of convergence of  $\zeta_{\mathfrak{g}_{137C}}^{\triangleleft}(s)$  is 4, with a simple pole at  $s = 4$ .

### 5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned} & \zeta_p(s) \zeta_p(s-1) \zeta_p(s-2) \zeta_p(s-3) \zeta_p(3s-4) \zeta_p(5s-5) \zeta_p(6s-9) \zeta_p(7s-4) \\ & \times \zeta_p(9s-6) \zeta_p(11s-10) \zeta_p(12s-10) \zeta_p(16s-11) W_1(p, p^{-s}) W_2(p, p^{-s}) \\ & \times W_3(p, p^{-s}) W_4(p, p^{-s}) W_5(p, p^{-s}) \end{aligned}$$

where

$$\begin{aligned} W_1(X, Y) &= 1 - X^9 Y^8, \\ W_2(X, Y) &= -1 + X^{10} Y^{11}, \\ W_3(X, Y) &= 1 - X^{10} Y^{12}, \\ W_4(X, Y) &= -1 + X^{11} Y^{17}, \\ W_5(X, Y) &= 1 - X^4 Y^8. \end{aligned}$$

The ghost is friendly.

### 6 Natural boundary

$\zeta_{\mathfrak{g}_{137C}}^{\triangleleft}(s)$  has a natural boundary at  $\Re(s) = 9/8$ , and is of type II.

### 7 Notes

This ideal zeta function is identical to that of  $\mathfrak{g}_{137D}$ , though the Lie rings themselves are non-isomorphic.