The zeta function of $\mathfrak{g}_{137B}$ counting ideals

1 Presentation

$\mathfrak{g}_{137B}$ has presentation

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \;\middle|\; \begin{array}{c} [x_1, x_2] = x_5, [x_1, x_5] = x_7, [x_2, x_4] = x_7, \\ [x_3, x_4] = x_6, [x_3, x_6] = x_7 \end{array} \right\rangle.$$

$\mathfrak{g}_{137B}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta^{\triangleleft}_{\mathfrak{g}_{137B},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)^2\zeta_p(5s-5)\zeta_p(7s-4) \\ & \times \zeta_p(8s-5)\zeta_p(9s-6)\zeta_p(12s-10)W(p, p^{-s}) \end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned} & 1 - X^4Y^5 - 2X^4Y^8 + X^5Y^8 + X^4Y^9 - 2X^5Y^9 + X^8Y^{12} - 2X^9Y^{12} + 3X^9Y^{13} \\ & - 2X^{10}Y^{13} + X^{10}Y^{14} + X^9Y^{17} + X^{14}Y^{17} + X^{13}Y^{20} - 2X^{13}Y^{21} + 3X^{14}Y^{21} \\ & - 2X^{14}Y^{22} + X^{15}Y^{22} - 2X^{18}Y^{25} + X^{19}Y^{25} + X^{18}Y^{26} - 2X^{19}Y^{26} \\ & - X^{19}Y^{29} + X^{23}Y^{34}. \end{aligned}$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{137B}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{137B},p}(s)\big|_{p \to p^{-1}} = -p^{21-17s}\zeta^{\triangleleft}_{\mathfrak{g}_{137B},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{g}_{137B}}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)^2\zeta_p(5s-5)\zeta_p(7s-4)\zeta_p(8s-5) \\ \times \zeta_p(9s-6)\zeta_p(12s-10)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})$$

where

$$\begin{aligned} W_1(X,Y) &= 1 + X^{14}Y^{17}, \\ W_2(X,Y) &= 1 + X^5Y^8, \\ W_3(X,Y) &= 1 + X^4Y^9. \end{aligned}$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{g}_{137B}}(s)$ has a natural boundary at $\Re(s) = 14/17$, and is of type III.

7 Notes

This ideal zeta function is identical to that of $M_3 \times_{\mathbb{Z}} M_3$, though the Lie rings themselves are non-isomorphic.