

# The zeta function of $\mathfrak{g}_{1357G}$ counting ideals

## 1 Presentation

$\mathfrak{g}_{1357G}$  has presentation

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \left| \begin{array}{l} [x_1, x_2] = x_3, [x_1, x_4] = x_6, [x_1, x_6] = x_7, \\ [x_2, x_3] = x_5, [x_2, x_5] = x_7 \end{array} \right. \right\rangle.$$

$\mathfrak{g}_{1357G}$  has nilpotency class 4.

## 2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta_{\mathfrak{g}_{1357G}, p}^{\triangleleft}(s) &= \zeta_p(s) \zeta_p(s-1) \zeta_p(s-2) \zeta_p(3s-4) \zeta_p(4s-3) \zeta_p(5s-5) \zeta_p(5s-6) \\ &\quad \times \zeta_p(6s-6) \zeta_p(7s-4) \zeta_p(7s-7) \zeta_p(8s-5) \zeta_p(9s-6) \\ &\quad \times \zeta_p(10s-9) \zeta_p(11s-8) \zeta_p(12s-10) \zeta_p(12s-11) W(p, p^{-s}) \end{aligned}$$

where  $W(X, Y)$  is

$$\begin{aligned} &1 + X^3Y^3 - X^3Y^5 - 2X^6Y^7 - 2X^4Y^8 + X^5Y^8 - X^7Y^8 + X^4Y^9 - 2X^5Y^9 \\ &- X^9Y^9 - X^7Y^{10} - X^{10}Y^{10} - X^7Y^{11} - X^8Y^{11} + X^9Y^{11} - X^{10}Y^{11} \\ &+ 3X^7Y^{12} - 3X^8Y^{12} - 2X^9Y^{12} + X^{10}Y^{12} - X^7Y^{13} + 3X^8Y^{13} + X^{10}Y^{13} \\ &- 3X^{11}Y^{13} + X^9Y^{14} + X^{10}Y^{14} - X^{11}Y^{14} + X^{13}Y^{14} + 5X^{11}Y^{15} - 2X^{12}Y^{15} \\ &- X^{14}Y^{15} + X^{11}Y^{16} + 5X^{12}Y^{16} - 2X^{14}Y^{16} + X^{16}Y^{16} + X^9Y^{17} - X^{12}Y^{17} \\ &+ X^{13}Y^{17} + 7X^{14}Y^{17} - X^{16}Y^{17} + X^{12}Y^{18} + 2X^{14}Y^{18} + 2X^{15}Y^{18} \\ &+ X^{16}Y^{18} + 2X^{12}Y^{19} - 4X^{14}Y^{19} + 6X^{15}Y^{19} + X^{16}Y^{19} + 3X^{17}Y^{19} \\ &- X^{12}Y^{20} + 4X^{13}Y^{20} - 3X^{15}Y^{20} + 2X^{16}Y^{20} + X^{17}Y^{20} + 3X^{18}Y^{20} \\ &- X^{12}Y^{21} - 2X^{13}Y^{21} + 4X^{14}Y^{21} + X^{15}Y^{21} - 2X^{17}Y^{21} + X^{18}Y^{21} \\ &+ 2X^{19}Y^{21} + X^{20}Y^{21} - 2X^{14}Y^{22} + X^{16}Y^{22} + X^{20}Y^{22} + X^{21}Y^{22} \\ &- 3X^{15}Y^{23} + X^{16}Y^{23} + 3X^{17}Y^{23} - 2X^{18}Y^{23} - X^{19}Y^{23} - X^{20}Y^{23} \\ &+ 2X^{21}Y^{23} - 5X^{16}Y^{24} - 3X^{18}Y^{24} + 6X^{19}Y^{24} - X^{20}Y^{24} - 6X^{21}Y^{24} \\ &+ X^{22}Y^{24} + X^{16}Y^{25} - 5X^{17}Y^{25} - X^{18}Y^{25} - 8X^{19}Y^{25} + 6X^{20}Y^{25} \\ &+ X^{21}Y^{25} - 2X^{22}Y^{25} + X^{17}Y^{26} - 2X^{18}Y^{26} - 4X^{19}Y^{26} - 6X^{20}Y^{26} \\ &+ X^{21}Y^{26} - X^{23}Y^{26} - 2X^{24}Y^{26} + X^{18}Y^{27} - X^{19}Y^{27} - 9X^{20}Y^{27} \\ &- 3X^{21}Y^{27} - 2X^{22}Y^{27} + X^{23}Y^{27} - X^{25}Y^{27} - X^{17}Y^{28} + 2X^{19}Y^{28} \end{aligned}$$

$$\begin{aligned}
& + X^{20}Y^{28} - 7X^{21}Y^{28} - 8X^{22}Y^{28} - 3X^{23}Y^{28} + 2X^{24}Y^{28} - X^{26}Y^{28} \\
& - X^{27}Y^{28} - X^{19}Y^{29} + 4X^{21}Y^{29} + X^{22}Y^{29} - 11X^{23}Y^{29} - 4X^{24}Y^{29} \\
& - 2X^{25}Y^{29} + X^{26}Y^{29} - 2X^{21}Y^{30} + 4X^{22}Y^{30} - X^{23}Y^{30} - 5X^{24}Y^{30} \\
& - 5X^{25}Y^{30} - 4X^{26}Y^{30} + X^{27}Y^{30} - X^{21}Y^{31} - X^{22}Y^{31} + 8X^{23}Y^{31} \\
& - X^{24}Y^{31} - 9X^{26}Y^{31} - X^{27}Y^{31} + X^{20}Y^{32} + X^{21}Y^{32} - 2X^{22}Y^{32} \\
& - 2X^{23}Y^{32} + 3X^{24}Y^{32} + 5X^{25}Y^{32} + 7X^{26}Y^{32} - 10X^{27}Y^{32} - X^{28}Y^{32} \\
& - 2X^{29}Y^{32} + X^{21}Y^{33} + 2X^{22}Y^{33} - 4X^{25}Y^{33} + 6X^{26}Y^{33} + 8X^{27}Y^{33} \\
& - 5X^{28}Y^{33} - X^{29}Y^{33} - X^{30}Y^{33} + 3X^{23}Y^{34} + X^{24}Y^{34} + 3X^{25}Y^{34} \\
& - X^{26}Y^{34} + X^{27}Y^{34} + 6X^{28}Y^{34} + 2X^{29}Y^{34} - 2X^{30}Y^{34} - X^{31}Y^{34} \\
& + 5X^{24}Y^{35} + X^{26}Y^{35} + X^{27}Y^{35} + 3X^{28}Y^{35} + 9X^{29}Y^{35} - X^{30}Y^{35} \\
& - X^{32}Y^{35} - 2X^{24}Y^{36} + 5X^{25}Y^{36} + 4X^{26}Y^{36} + 10X^{27}Y^{36} - 8X^{28}Y^{36} \\
& - 5X^{29}Y^{36} + 12X^{30}Y^{36} + X^{31}Y^{36} + 4X^{32}Y^{36} - X^{33}Y^{36} - X^{25}Y^{37} \\
& + 2X^{26}Y^{37} + 15X^{28}Y^{37} - X^{29}Y^{37} - 2X^{30}Y^{37} + 4X^{31}Y^{37} + X^{32}Y^{37} \\
& + 3X^{33}Y^{37} - X^{26}Y^{38} - X^{27}Y^{38} + 2X^{28}Y^{38} + 13X^{29}Y^{38} + 2X^{30}Y^{38} \\
& + X^{31}Y^{38} - X^{32}Y^{38} + 4X^{33}Y^{38} + X^{34}Y^{38} + X^{35}Y^{38} - 2X^{27}Y^{39} - X^{28}Y^{39} \\
& + 2X^{29}Y^{39} + 9X^{30}Y^{39} + 5X^{31}Y^{39} - X^{33}Y^{39} + 3X^{34}Y^{39} + X^{36}Y^{39} \\
& + X^{27}Y^{40} - X^{28}Y^{40} - 6X^{29}Y^{40} - 5X^{30}Y^{40} + 13X^{31}Y^{40} + 12X^{32}Y^{40} \\
& - 3X^{33}Y^{40} - X^{34}Y^{40} + X^{35}Y^{40} + X^{36}Y^{40} + X^{37}Y^{40} + X^{29}Y^{41} - 2X^{30}Y^{41} \\
& - 10X^{31}Y^{41} + 8X^{33}Y^{41} + 9X^{34}Y^{41} - 2X^{35}Y^{41} - X^{36}Y^{41} + X^{30}Y^{42} \\
& - 5X^{31}Y^{42} - 4X^{32}Y^{42} - X^{33}Y^{42} + 3X^{34}Y^{42} + 7X^{35}Y^{42} - 2X^{36}Y^{42} \\
& + X^{37}Y^{42} - X^{29}Y^{43} + 2X^{31}Y^{43} - 5X^{32}Y^{43} - 6X^{33}Y^{43} - 12X^{34}Y^{43} \\
& + 6X^{35}Y^{43} + 6X^{36}Y^{43} - X^{39}Y^{43} - X^{30}Y^{44} - X^{31}Y^{44} + X^{32}Y^{44} \\
& + 2X^{33}Y^{44} - 6X^{34}Y^{44} - 18X^{35}Y^{44} + 4X^{36}Y^{44} + 2X^{37}Y^{44} + 3X^{38}Y^{44} \\
& - X^{31}Y^{45} - 3X^{32}Y^{45} - 3X^{33}Y^{45} + 6X^{34}Y^{45} + X^{35}Y^{45} - 15X^{36}Y^{45} \\
& - 8X^{37}Y^{45} - X^{38}Y^{45} + 5X^{39}Y^{45} - X^{40}Y^{45} - 2X^{33}Y^{46} - 4X^{34}Y^{46} \\
& - X^{36}Y^{46} - 6X^{37}Y^{46} - 6X^{38}Y^{46} - 2X^{39}Y^{46} + X^{40}Y^{46} - 3X^{34}Y^{47} \\
& - 7X^{35}Y^{47} + 4X^{36}Y^{47} + 3X^{37}Y^{47} - 12X^{38}Y^{47} - 5X^{39}Y^{47} - 5X^{40}Y^{47} \\
& + 2X^{41}Y^{47} + 2X^{35}Y^{48} - 12X^{36}Y^{48} + 9X^{38}Y^{48} - 6X^{39}Y^{48} - 10X^{41}Y^{48} \\
& + X^{42}Y^{48} + X^{35}Y^{49} + 4X^{36}Y^{49} - 10X^{37}Y^{49} - 5X^{38}Y^{49} + X^{39}Y^{49} \\
& + X^{41}Y^{49} - 7X^{42}Y^{49} - X^{43}Y^{49} + X^{36}Y^{50} + 4X^{37}Y^{50} - 4X^{38}Y^{50} \\
& - 5X^{39}Y^{50} - 5X^{40}Y^{50} + 6X^{41}Y^{50} - X^{42}Y^{50} - 2X^{43}Y^{50} - 2X^{44}Y^{50} \\
& + X^{37}Y^{51} + 6X^{38}Y^{51} - 2X^{39}Y^{51} - 6X^{40}Y^{51} - 3X^{41}Y^{51} + 5X^{42}Y^{51} \\
& - X^{44}Y^{51} - 2X^{45}Y^{51} + X^{37}Y^{52} + 9X^{39}Y^{52} + 4X^{40}Y^{52} - 7X^{41}Y^{52} \\
& - 7X^{42}Y^{52} + X^{43}Y^{52} + 5X^{44}Y^{52} - X^{45}Y^{52} - X^{46}Y^{52} - 2X^{39}Y^{53}
\end{aligned}$$

$$\begin{aligned}
& + 5X^{40}Y^{53} + 8X^{41}Y^{53} + 4X^{42}Y^{53} - 8X^{43}Y^{53} - 2X^{44}Y^{53} + 3X^{45}Y^{53} \\
& + X^{40}Y^{54} + 3X^{41}Y^{54} + 6X^{42}Y^{54} + 7X^{43}Y^{54} - 5X^{44}Y^{54} - X^{45}Y^{54} \\
& + X^{46}Y^{54} + 2X^{47}Y^{54} - X^{48}Y^{54} + X^{39}Y^{55} + X^{40}Y^{55} - X^{41}Y^{55} + X^{42}Y^{55} \\
& + 10X^{43}Y^{55} + 5X^{44}Y^{55} - X^{45}Y^{55} - 2X^{46}Y^{55} - 2X^{47}Y^{55} + 3X^{48}Y^{55} \\
& + X^{41}Y^{56} - 2X^{42}Y^{56} - X^{43}Y^{56} + 8X^{44}Y^{56} + 7X^{45}Y^{56} + 5X^{46}Y^{56} \\
& - 4X^{47}Y^{56} - X^{48}Y^{56} + X^{49}Y^{56} + 3X^{42}Y^{57} - 4X^{44}Y^{57} + 2X^{45}Y^{57} \\
& + 9X^{46}Y^{57} + 7X^{47}Y^{57} - 2X^{48}Y^{57} + 3X^{44}Y^{58} - 2X^{46}Y^{58} + 4X^{47}Y^{58} \\
& + 5X^{48}Y^{58} + X^{44}Y^{59} + X^{45}Y^{59} - 3X^{46}Y^{59} + 2X^{48}Y^{59} + 7X^{49}Y^{59} \\
& - X^{44}Y^{60} + X^{45}Y^{60} + 2X^{46}Y^{60} - X^{48}Y^{60} - 6X^{49}Y^{60} + 7X^{50}Y^{60} \\
& + X^{51}Y^{60} - X^{45}Y^{61} + 3X^{48}Y^{61} - X^{49}Y^{61} - 5X^{50}Y^{61} + 4X^{51}Y^{61} \\
& + X^{52}Y^{61} - 2X^{46}Y^{62} - X^{47}Y^{62} - 3X^{48}Y^{62} + 4X^{49}Y^{62} - 4X^{51}Y^{62} \\
& - X^{52}Y^{62} + 2X^{53}Y^{62} - 2X^{47}Y^{63} + X^{48}Y^{63} - 2X^{50}Y^{63} + 2X^{51}Y^{63} \\
& - 3X^{52}Y^{63} - 2X^{53}Y^{63} + 2X^{54}Y^{63} + X^{47}Y^{64} - 3X^{48}Y^{64} - 5X^{49}Y^{64} \\
& - X^{50}Y^{64} + X^{51}Y^{64} + 3X^{52}Y^{64} - 2X^{53}Y^{64} - 2X^{54}Y^{64} - 6X^{51}Y^{65} \\
& + X^{53}Y^{65} - X^{54}Y^{65} - X^{51}Y^{66} - 6X^{52}Y^{66} - X^{56}Y^{66} - X^{50}Y^{67} - 2X^{53}Y^{67} \\
& - X^{54}Y^{67} - X^{55}Y^{67} + X^{56}Y^{67} - X^{57}Y^{67} + 2X^{52}Y^{68} - 4X^{54}Y^{68} \\
& - 3X^{55}Y^{68} + X^{56}Y^{68} - X^{53}Y^{69} + 3X^{54}Y^{69} - 3X^{56}Y^{69} - X^{57}Y^{69} \\
& - X^{58}Y^{70} + 2X^{55}Y^{71} + X^{57}Y^{71} - X^{59}Y^{71} + 2X^{58}Y^{72} - X^{59}Y^{72} + X^{56}Y^{73} \\
& - X^{57}Y^{73} + 2X^{59}Y^{73} + 2X^{57}Y^{74} + X^{60}Y^{74} - X^{60}Y^{75} + 2X^{61}Y^{75} \\
& + X^{59}Y^{76} + X^{60}Y^{76} - X^{61}Y^{76} + X^{62}Y^{76} + X^{60}Y^{77} + X^{63}Y^{78} - X^{63}Y^{81} \\
& - X^{66}Y^{83}.
\end{aligned}$$

$\zeta_{\mathfrak{g}_{1357G}}^{\triangleleft}(s)$  is uniform.

### 3 Functional equation

The local zeta function satisfies no functional equation.

### 4 Abscissa of convergence and order of pole

The abscissa of convergence of  $\zeta_{\mathfrak{g}_{1357G}}^{\triangleleft}(s)$  is 3, with a simple pole at  $s = 3$ .

## 5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-4)\zeta_p(4s-3)\zeta_p(5s-5)\zeta_p(5s-6)\zeta_p(6s-6) \\ & \times \zeta_p(7s-4)\zeta_p(7s-7)\zeta_p(8s-5)\zeta_p(9s-6)\zeta_p(10s-9)\zeta_p(11s-8)\zeta_p(12s-10) \\ & \times \zeta_p(12s-11)W_1(p, p^{-s})W_2(p, p^{-s})W_3(p, p^{-s})W_4(p, p^{-s})W_5(p, p^{-s}) \\ & \times W_6(p, p^{-s}) \end{aligned}$$

where

$$\begin{aligned} W_1(X, Y) &= 1 + X^3Y^3 - X^9Y^9 - X^{10}Y^{10} + X^{16}Y^{16}, \\ W_2(X, Y) &= 1 - X^{11}Y^{12}, \\ W_3(X, Y) &= -1 + X^{10}Y^{12}, \\ W_4(X, Y) &= 1 - X^{11}Y^{14}, \\ W_5(X, Y) &= -1 - X^9Y^{13}, \\ W_6(X, Y) &= -1 - X^9Y^{16}. \end{aligned}$$

The ghost is unfriendly.

## 6 Natural boundary

$\zeta_{\mathfrak{g}_{1357G}}^{\triangleleft}(s)$  has a natural boundary at  $\Re(s) = 1$ , and is of type I.

## 7 Notes

This ideal zeta function is identical to that of  $\mathfrak{g}_{1357H}$ , though the Lie rings themselves are non-isomorphic.